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SPACES OF
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HOLOMORPHIC IN
CONVEX DOMAINS OF
 (C^n)**

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Abstract

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MATHEMATICS

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GENERAL FORM OF A LINEAR CONTINUOUS FUNCTIONAL IN SPACES OF FUNCTIONS HOLOMORPHIC IN CONVEX DOMAINS OF C^n

(Presented by Academician M. A. Lavrent'ev, 17-V 1965)

1. Let D be a domain in the complex space C^n of complex variables $(z_1, z_2, \dots, z_n) = z$; $A(D)$ a linear topological space of functions $f(z)$ holomorphic in the domain D ; $A(\overline{D})$ a linear topological space of functions holomorphic in the closed domain \overline{D} . The topology in $A(D)$ and $A(\overline{D})$ is introduced in the generally accepted manner; here the space $A(D)$ is a countably normed space (see ⁽¹⁾, p. 29), while $A(\overline{D})$ is an inductive limit (union) of countably normed spaces (see ⁽¹⁾, p. 89; ^(2,3)). A sequence $f_m(z)$, $m = 1, 2, \dots$, converges to $f(z)$ in the topology of the space $A(D)$ if this sequence converges to the function $f(z)$ uniformly on every closed set $M \subset D$. A sequence $f_m(z)$, $m = 1, 2, \dots$, converges to $f(z)$ in the topology of the space $A(\overline{D})$ if there exists a domain $Q \supset \overline{D}$ in which all the functions $f_m(z)$, $m = 1, 2, \dots$, are holomorphic, and $\lim_{m \rightarrow \infty} f_m(z) = f(z)$ in the sense of the topology of the space $A(Q)$.

One of the most important questions arising in the study of the spaces $A(D)$ and $A(\overline{D})$ is the question of the general form of a linear continuous functional in these spaces. In the case $n = 1$ an exhaustive answer to this question has been given (see ⁽⁴⁻⁶⁾); moreover it turned out that the space $A^*(D)$ is isomorphic to $A(C^1 \setminus D)$, and $A^*(\overline{D})$ is isomorphic to $A(C^1 \setminus \overline{D})$, where $A^*(D)$ (respectively $A^*(\overline{D})$) is the space conjugate to $A(D)$ (respectively to $A(\overline{D})$). Similar results are valid ⁽⁷⁾ for polycylindrical domains in C^n , $n > 1$ (i.e., for topological products of plane domains). For domains $D \subset C^n$, $n > 1$, that are not polycylindrical, analogous assertions cannot be obtained.

The problem of describing the general form of a linear continuous functional in the space $A(D)$ (in $A(\overline{D})$) was solved (see ⁽⁸⁻¹⁴⁾) by other methods for certain classes of domains D (n -circular, (p, q) -circular, semicircular, tubular). In the present note the indicated problem is solved for convex domains $D \subset C^n$ by means of the integral representation obtained in ⁽¹⁵⁾. For these domains D the

concept of the conjugate set \widetilde{D} is introduced; in a certain sense it plays the same role as the exterior of a domain does when $n = 1$.

2. Let D be a convex domain, $(0, 0, \dots, 0) \in D$. Put

$$\widetilde{D} = \{w : w_1 z_1 + \dots + w_n z_n \neq 1, z \in D\}, \quad \widetilde{\widetilde{D}} = \{w : w_1 z_1 + \dots + w_n z_n \neq 1, z \in \overline{D}\}.$$

Then:

1°. \widetilde{D} is a closed star-shaped bounded set, and D is a star-shaped bounded domain.

2°. If $D_1 \subset D$, then $\widetilde{D}_1 \supset \widetilde{D}$, $\widetilde{\widetilde{D}}_1 \supset \widetilde{\widetilde{D}}$.

3°. Denote by D_r , $r > 0$, the homothety $D_r = r \cdot D$ of the domain D . Then

$$(\widetilde{D}_r) = (\widetilde{D})_{1/r}.$$

Analogously,

$$(\widetilde{\widetilde{D}}_r) = (\widetilde{\widetilde{D}})_{1/r}.$$

4°. If the domain D is circular, then \widetilde{D} and $\widetilde{\widetilde{D}}$ are circular convex sets.*

3. The domain D can be represented in the form $D = \lim_{m \rightarrow \infty} D_m$, where $\overline{D}_m \subset D_{m+1}$, $(0, 0, \dots, 0) \in D_m$,

$$D_m = \{z : \Phi_m(z, \bar{z}) < 0\}, \quad m = 1, 2, \dots, \quad (1)$$

the domains D_m are bounded, the functions Φ_m are twice continuously differentiable and convex in some neighborhoods of \overline{D}_m . Denote by ∂D_m the boundary of the domain D_m , and by $\partial \widetilde{D}_m$ the boundary of \widetilde{D}_m . Note that $\widetilde{D} = \lim_{m \rightarrow \infty} \widetilde{D}_m$, $\widetilde{\widetilde{D}}_{m+1} \subset (\widetilde{D}_m \setminus \partial \widetilde{D}_m)$, $m = 1, 2, \dots$. Let

$$\omega(\Phi) = (n-1)! \left(\sum_{k=1}^n \delta_k d\bar{\xi}[k] \right) \wedge d\xi / (2\pi i)^n (\xi_1 \Phi'_{\xi_1} + \dots + \xi_n \Phi'_{\xi_n})^n,$$

where

$$d\bar{\xi}[k] = d\bar{\xi}_1 \wedge \dots \wedge d\bar{\xi}_{k-1} \wedge d\bar{\xi}_{k+1} \wedge \dots \wedge d\bar{\xi}_n; \quad d\xi = d\xi_1 \wedge \dots \wedge d\xi_n;$$

\wedge denotes exterior multiplication; $\Phi = \Phi(\xi, \bar{\xi})$ is a twice continuously differentiable function,

$$\delta_k = \begin{vmatrix} \Phi'_{\xi_1} & \Phi'_{\xi_2} & \dots & \Phi'_{\xi_n} \\ \Phi''_{\xi_1 \bar{\xi}_1} & \Phi''_{\xi_2 \bar{\xi}_1} & \dots & \Phi''_{\xi_n \bar{\xi}_1} \\ \vdots & \vdots & & \vdots \\ \Phi''_{\xi_1 \bar{\xi}_{k-1}} & \Phi''_{\xi_2 \bar{\xi}_{k-1}} & \dots & \Phi''_{\xi_n \bar{\xi}_{k-1}} \\ \Phi''_{\xi_1 \bar{\xi}_{k+1}} & \Phi''_{\xi_2 \bar{\xi}_{k+1}} & \dots & \Phi''_{\xi_n \bar{\xi}_{k+1}} \\ \vdots & \vdots & & \vdots \\ \Phi''_{\xi_1 \bar{\xi}_n} & \Phi''_{\xi_2 \bar{\xi}_n} & \dots & \Phi''_{\xi_n \bar{\xi}_n} \end{vmatrix}.$$

Further, let $\tau(\Phi) = (\tau_1(\Phi), \dots, \tau_n(\Phi))$,

$$\tau_i(\Phi) = \Phi'_{\xi_i} (\xi_1 \Phi'_{\xi_1} + \dots + \xi_n \Phi'_{\xi_n})^{-1}, \quad i = 1, 2, \dots, n.$$

Theorem 1. *Every continuous linear functional $F(f)$ in the space $A(D)$ has the form*

$$F(f) = F_\varphi(f) = \int_{\partial \tilde{D}_m} f(\xi) \varphi(\tau(\Phi_m)) \omega(\Phi_m), \quad (2)$$

where the function φ is holomorphic on the set \tilde{D} ; m depends only on φ and is such that φ is holomorphic on the set $\tilde{D}_m \supset \tilde{D}$. This correspondence between continuous linear functionals $F(f)$ in the space $A(D)$ and functions $\varphi \in A(\tilde{D})$ is an isomorphism of the linear topological spaces $A^*(D)$ and $A(\tilde{D})$.

Proof. 1) First of all, note that, by the convexity of Φ_m , all first-order derivatives of Φ_m do not vanish simultaneously at points of the boundary ∂D_m of the domain D_m . At each point $\xi \in \partial D_m$ there exists a tangent complex $(n - 1)$ -dimensional analytic plane

$$\begin{aligned} \{z : (\xi_1 - z_1) \Phi'_{m\xi_1} + \dots + (\xi_n - z_n) \Phi'_{m\xi_n} = \\ = 0\} = \{z : z_1 \tau_1(\Phi_m) + \dots + z_n \tau_n(\Phi_m) = 1\} \end{aligned}$$

$$(\xi_1 \Phi'_{m\xi_1} + \dots + \xi_n \Phi'_{m\xi_n} \neq 0, \text{ since } (0, 0, \dots, 0) \in D_m),$$

which does not intersect the convex domain D_m ; therefore the point $\tau(\Phi_m) \in \tilde{D}_m$.

2) Let the functional $F(f) \in A^*(D)$; then there exists an m_0 such that

* If the domain D is not circular, then one cannot guarantee the convexity of \widetilde{D} or $\widetilde{\widetilde{D}}$. For example, if $D = \{z : |\operatorname{Re} z| + |\operatorname{Im} z| < 1\} \subset C^1$, then the set $\widetilde{D} = \{w : (|\operatorname{Re} w| - 1/2)^2 + (|\operatorname{Im} w| - 1/2)^2 \leq 1/2\}$ is star-shaped, but not convex.

$F(f)$ is bounded with respect to the norm $\|f\|_{m_0} = \max_{\overline{D}_{m_0}} |f|$ (see (1), p. 49). Using the integral representation

$$f(z) = \int_{\partial D_m} \frac{f(\zeta) \omega(\Phi_m)}{[1 - z_1 \tau_1(\Phi_m) - \dots - z_n \tau_n(\Phi_m)]^n}, \quad (3)$$

$z \in D_m$, $\zeta \in \partial D_m$, we obtain, for $m > m_0$, formula (2), where $\varphi(\tau(\Phi_m)) = F([1 - z_1 \tau_1(\Phi_m) - \dots - z_n \tau_n(\Phi_m)]^{-n})$, the function φ is holomorphic on the set \widetilde{D}_m , and the integral on the right-hand side of (2) does not depend on the choice of $m > m_0$.

Conversely, let $\varphi \in A(\widetilde{D})$; then there is an m such that $\varphi \in A(\widetilde{D}_m)$. Then, obviously, formula (2) defines a linear continuous functional $F(f)$ in the space $A(D)$.

- 3) From the boundedness of the domain D_m it follows that there exists an $r > 0$ such that D_m is contained in the hypersphere $I_r = \{z : |z_1|^2 + \dots + |z_n|^2 < r^2\}$. It can be shown that equality (2), considered only for $f(z) \in A(\overline{I}_r)$, is transformed into the form

$$\begin{aligned} F_\varphi(f) &= \frac{(n-1)!}{(2\pi i)^n r^{2n}} \int_{\partial I_r} f(\zeta) \varphi\left(\frac{\bar{\zeta}}{r^2}\right) \left(\sum_{k=1}^n (-1)^{k-1} \bar{\zeta}_k d\bar{\zeta}[k]\right) \wedge d\zeta = \\ &= \frac{(n-1)! r^{2n}}{(2\pi i)^n} \int_{\partial I_{1/r}} f(\bar{\eta} r^2) \varphi(\eta) \sum_{k=1}^n (-1)^{k-1} \bar{\eta}_k d\bar{\eta}[k] \wedge d\eta, \end{aligned} \quad (4)$$

where $\zeta \in \partial I_r$, $\eta \in \partial I_{1/r}$, $\bar{I}_{1/r} = \tilde{I}_r$. Put $f_w(z) = (1 - w_1 z_1 - \dots - w_n z_n)^{-n}$, where w is a fixed point of $I_{1/r}$. Applying (3) to the domain $I_{1/r}$ and to the function φ (instead of the function f) and (4), we obtain

$$F_\varphi(f_w) = \varphi(w) \quad (5)$$

for all $w \in I_{1/r}$. On the other hand, it follows from (2) that the function $\psi(w) = F_\varphi(f_w)$ is holomorphic in w in the domain \widetilde{D}_m ; therefore $\psi(w) = \varphi(w)$ for $w \in \widetilde{D}_m$, and hence formula (5) is also valid for $w \in \widetilde{D}_m$.

- 4) In 2) a correspondence was established between functionals $F(f) \in A^*(D)$ and functions $\varphi \in A(\widetilde{D})$. We shall show that this correspondence is one-to-one. Let $\varphi_1, \varphi_2 \in A(\widetilde{D})$; choose m so that $\varphi_1, \varphi_2 \in A(\widetilde{D}_m)$. If $\varphi_1 = \varphi_2$,

then from (2) we find $F_{\varphi_1}(f) = F_{\varphi_2}(f)$. Conversely, if $F_{\varphi_1}(f) = F_{\varphi_2}(f)$, then from (5) it follows that $\varphi_1 = \varphi_2$ in the domain $\widetilde{D}_m \subset \widetilde{D}_m$, i.e. $\varphi_1 = \varphi_2$ also on the set \widetilde{D}_m .

From the one-to-one character of the correspondence under consideration and formulas (2) and (5), we obtain that the spaces $A^*(D)$ and $A(\widetilde{D})$ are algebraically isomorphic. It remains to establish that this isomorphism is also topological. The spaces $A^*(D)$ and $A(\widetilde{D})$ are, as can be shown, spaces (LN^*) (see (2, 3)); therefore it suffices to prove that a sequence φ_k , $k = 1, 2, \dots$, converges in the topology of the space $A(\widetilde{D})$ to a function φ if and only if the sequence of functionals F_{φ_k} , $k = 1, 2, \dots$, converges in the sense of the topology of the space $A^*(D)$ to the functional F_φ .

- 5) Let $\varphi = \lim_{k \rightarrow \infty} \varphi_k$ in the topology of the space $A(\widetilde{D})$; then there exists an m such that the indicated convergence is uniform on the closed set \widetilde{D}_m , and, by virtue of (2), $\lim_{k \rightarrow \infty} F_{\varphi_k}(f) = F_\varphi(f)$ for all $f \in A(D)$, i.e. the sequence of functionals F_{φ_k} converges weakly to the functional F_φ . By Montel's theorem (see (16), p. 195) it follows that the space $A(D)$ is perfect ((1), p. 73); therefore $\lim_{k \rightarrow \infty} F_{\varphi_k} = F_\varphi$ with respect to the topology of the space $A^*(D)$ ((1), p. 77).

Conversely, let $\lim_{k \rightarrow \infty} F_{\varphi_k} = F_\varphi$ in the topology of the space $A^*(D)$. There exists an m such that all $F_{\varphi_k} \in B^*(D_m)$ and $\lim_{k \rightarrow \infty} F_{\varphi_k} = F_\varphi$ in the norm of the space

spaces $B^*(D_m)$, where $B^*(D_m)$ is the space conjugate to the Banach space $B(D_m)$ of functions holomorphic in the domain D_m , continuous in the closed domain \overline{D}_m , with norm $\|f\|_m = \max_{\overline{D}_m} |f|$ (see (1), p. 78). Repeating the arguments of 2), we obtain that all φ_k , $k = 1, 2, \dots$, are holomorphic on the set \widetilde{D}_{m+1} . Moreover,

$$\lim_{k \rightarrow \infty} F_{\varphi_k}(f_w) = F_\varphi(f_w)$$

uniformly for $w \in \widetilde{D}_{m+1}$. Consequently, by virtue of (5),

$$\lim_{k \rightarrow \infty} \varphi_k(w) = \varphi(w),$$

and this convergence is uniform on the closed set $\widetilde{D}_{m+1} \supset \widetilde{D}$.

4. A bounded domain D can be represented in the form

$$D = \lim_{m \rightarrow \infty} D_m,$$

$\overline{D}_{m+1} \subset D_m$, where the domains D_m have the form (1) under the same restrictions as in item 3. Then, analogously to Theorem 1, the following can be established.*

Theorem 2. Let the domain D be bounded. Every linear continuous functional $F(f)$ on the space $A(\overline{D})$ has the form (2), where the function φ is holomorphic

in the domain \tilde{D} , m depends on f and is such that f is holomorphic on the set $\overline{D}_m \supset \overline{D}$. This correspondence between linear continuous functionals on the space $A(\overline{D})$ and functions $\varphi \in A(\tilde{D})$ is an isomorphism of linear topological spaces $A^*(\overline{D})$ and $A(\tilde{D})$.

5. If the domain $D = \{z : \Phi(z, \bar{z}) < 0\}$ is bounded, the function Φ is twice continuously differentiable and convex in some neighborhood of \overline{D} , then instead of the domains D_m in Theorem 1 (in Theorem 2) one may consider homothetic domains D_r , $r < 1$ ($r > 1$) or domains

$$D(\rho) = \{z : \Phi(z, \bar{z}) < \rho\}, \quad \rho < 0 \quad (\rho > 0).$$

Then we obtain

Corollary 1. If, for any nonnegative integers k_1, k_2, \dots, k_n ,

$$\int_{\partial D} \zeta_1^{k_1} \dots \zeta_n^{k_n} \varphi(\tau(\Phi)) \omega(\Phi) = 0,$$

where $\zeta \in \partial D$, and the function φ is holomorphic on the set \tilde{D} , then $\varphi = 0$.

Corollary 2. In order that, for every function $f(z)$ holomorphic in the closed domain \overline{D} , the formula

$$f(z) = \int_{\partial D} f(\zeta) \Psi(z, \bar{z}, \tau(\Phi)) \omega(\Phi)$$

hold, where $\zeta \in \partial D$, $z \in D$, and $\Psi(z, \bar{z}, \tau(\Phi))$ is holomorphic in τ on the set \tilde{D} for each $z \in D$, it is necessary and sufficient that

$$\Psi(z, \bar{z}, \tau(\Phi)) = \frac{1}{[1 - z_1 \tau_1(\Phi) - \dots - z_n \tau_n(\Phi)]^n}.$$

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* Theorem 2 is obtained also as a consequence of Martineau' s results ⁽¹⁷⁾.

Note: Figure translations are in progress. See original paper for figures.

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