THE DECISION PROBLEM

MATHEMATICS

1966

SovietRxiv

Abstract

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UDC 517.11

MATHEMATICS

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THE DECISION PROBLEM

FOR THE NARROW PREDICATE CALCULUS

(Presented by Academician A. I. Mal' cev on 1 XI 1965)

Let Φ be a class of formulas. We shall say that Φ is **decidable** if the problems of recognizing satisfiability and finite satisfiability of formulas of the class Φ are algorithmically decidable. We shall say that Φ is a **reduction class** if there exists an algorithm which, to every formula α of the narrow predicate calculus with equality, assigns such a formula $\beta \in \Phi$ that α and β are simultaneously satisfiable or unsatisfiable and simultaneously satisfiable or unsatisfiable on finite models.

In ([1]) a summary is given of results on the decision problem. The corollaries formulated below follow from these results and from the theorem of the present article. Here F everywhere is the symbol of a binary predicate, and M denotes a quantifier-free formula.

Theorem. The class of formulas of the NPC without equality of the form

$$\forall x \exists u \forall y \exists z_1 \dots z_n M(F; x, u, y, z_1, \dots, z_n) \tag{1}$$

is a reduction class.

Let Π be some set of words in the alphabet $\{\forall, \exists\}$, and let σ be a collection of predicate symbols. By $\Phi(\Pi, \sigma)$ we denote the class of all formulas of the NPC without equality of the form

$$Q_1 x_1 \dots Q_{nx} n M(\sigma; x_1, \dots, x_n),$$

where $Q_1 \dots Q_n \in \Pi$. By $\Phi^*(\Pi, \sigma)$ we denote the corresponding class of formulas of the NPC with equality.

Corollary 1. Either $\Phi(\Pi, \sigma)$ is decidable, or it is a reduction class. The same also applies to $\Phi^*(\Pi, \sigma)$.

Corollary 2. $\Phi(\Pi, \sigma)$ is decidable if and only if $\Phi^*(\Pi, \sigma)$ is decidable.

Corollary 3. Let

$$\Pi_1 = \{\exists^m \forall^n \mid m, n = 0, 1, ...\}, \qquad \Pi_2 = \{\exists^m \forall^2 \exists^n \mid m, n = 0, 1, ...\}.$$

$\Phi(\Pi, \sigma)$ is decidable if and only if at least one of the following three possibilities holds:

a) σ contains only symbols of unary predicates;

b) $\Pi \subseteq \Pi_1 \cup \Pi_2$;

c) σ and $\Pi \setminus (\Pi_1 \cup \Pi_2)$ are finite.

Corollary 4. If $\Phi^*(\Pi, \sigma)$ is decidable, then, with the exception of a finite number of formulas, satisfiability for formulas of $\Phi^*(\Pi, \sigma)$ coincides with satisfiability on finite models.

Remark. The indicated corollaries for the case of infinite σ are given in the works of a group of American mathematicians headed by Wang Hao.

We outline the proof of the theorem. Let $i = \varepsilon(i) + 2\delta(i)$ be the binary notation, $i = 0, 1, 2, 3$, and let

$$\varphi = \forall x \exists u \forall y \exists z_1 \dots z_n (A_0.A_1, \dots A_l.M)$$

be a formula of the form (1). Instead of Fab we shall write simply ab.

$$A_0 = (\neg xx.yy \supset \neg uu.(xu \sim xy).(ux \sim yx)).z_1 z_1.$$

Put $f_0^i x \sim \neg xx.(-1)^{\varepsilon(i)} xu.(-1)^{\delta(i)} Ux$. In view of A_0, $f_0^i u$ may be regarded as an abbreviation for $\neg uu.(-1)^{\varepsilon(i)} uz_1.(-1)^{\delta(i)} z_1 u$.

$$A_1 = [\neg xx.\neg f_0^3 x.f_0^3 y \supset \neg uu.\neg f_0^3 u. \bigwedge_{i=0}^{2} (f_0^i u \sim (-1)^{\varepsilon(i)} xy.(-1)^{\delta(i)} yx)].f_0^3 z_2.$$

Put, for $i = 0, 1, 2$, $f_1^i x \sim \neg xx.\neg f_0^3 x.f_0^i u$. Then $f_1^i u$, in view of A_1, may be regarded as an abbreviation for $\neg uu.\neg f_0^3 u.(-1)^{\varepsilon(i)} uz_2.(-1)^{\delta(i)} z_2 u$, and so on, so that in M we can use an already larger number of unary predicates. Then the theorem from [2] is applied.

The author expresses his gratitude to Academician A. I. Mal' cev for posing the problem.

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Received
28 X 1965

CITED LITERATURE

[1] V. F. Kostyrko, *Algebra and Logic*, 3, 5-6, 45 (1964). [2] Yu. Sh. Gurevich, DAN, 166, No. 5 (1966).

Note: Figure translations are in progress. See original paper for figures.