DIRECT AND INVERSE SCATTERING PROBLEMS FOR AN EQUATION IN PARTIAL DIFFERENCES

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Abstract

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MATHEMATICS

M. S. ESKINA

DIRECT AND INVERSE SCATTERING PROBLEMS FOR AN EQUATION IN PARTIAL DIFFERENCES

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1. Let $R = (R_1, R_2, \dots, R_n)$ be an integer vector of the n-dimensional space R^n; let $\mathfrak{A}$ be a set of integer points of R^n, symmetric with respect to the origin and located inside the octahedron

$$|R_1| + |R_2| + \dots + |R_n| \le m,$$

and suppose that some two opposite vertices of the octahedron (for example, the points $(m, 0, \dots, 0)$ and $(-m, 0, \dots, 0)$) belong to $\mathfrak{A}$. On finite sequences u_R consider the difference expression

$$(\mathcal{L}_0 u)_R = \sum_{R' \in \mathfrak{A}} a_{R'} u_{R-R'}, \tag{1}$$

where the coefficients a_R satisfy the relation

$$a_R = \bar{a}_{-R} \qquad (a_R \ne 0 \quad \text{for } R \in \mathfrak{A}). \tag{2}$$

The closure in $l_2(R^n)$ of the operator constructed from the difference expression (1) gives a bounded self-adjoint operator L_0.

Let $\xi = (\xi_1, \dots, \xi_n)$, where $0 \le \xi_i \le 2\pi$; let Ω be the set of such ξ. Denote

$$P(\xi) = \sum_{R \in \mathfrak{A}} a_R e^{i(\xi, R)}, \qquad \alpha = \min_{\xi \in \Omega} P(\xi), \qquad \beta = \max_{\xi \in \Omega} P(\xi). \tag{3}$$

By virtue of (2), $\operatorname{Im} P(\xi) = 0$.

From the unitary equivalence of the operator L_0 to the operator of multiplication by $P(\xi)$ in $L_2(\Omega)$, it follows that the discrete spectrum of L_0 is absent, while the continuous spectrum fills the interval $[\alpha, \beta]$.

Let $\lambda \in (\alpha, \beta)$ and $\varepsilon > 0$. For $z = \lambda + i\varepsilon$ the operator L_0 has a resolvent with kernel

$$H_0(R - R_0; \lambda + i\varepsilon) = \frac{1}{(2\pi)^n} \int_\Omega \frac{e^{-i(\xi, R-R_0)}\, d\xi}{P(\xi) - \lambda - i\varepsilon}. \tag{4}$$

Lemma 1. Suppose that $\lambda \in (\alpha, \beta)$ is such that the surface $P(\xi) = \lambda$ has no singular points, i.e. $\operatorname{grad} P(\xi) \neq 0$ on the surface $P(\xi) = \lambda$. Then the function $H_0(R; z)$, for each fixed R, is continuous as a function of $(z = \lambda + i\varepsilon,\ \varepsilon \geq 0)$, so that there exists

$$\lim_{\varepsilon \to 0} H_0(R; \lambda + i\varepsilon) = H_0(R; \lambda),$$

and moreover

$$H_0(R; \lambda) = \frac{1}{(2\pi)^n} \operatorname{V.p.} \int_\Omega \frac{e^{-i(\xi, R)}\, d\xi}{P(\xi) - \lambda} - \frac{i\pi}{(2\pi)^n} \int_{P(\xi') = \lambda} \frac{e^{-i(\xi', R)}\, d\xi'}{|\operatorname{grad} P(\xi')|} = \frac{1}{(2\pi)^n} \int_\Omega \frac{e^{-i(\xi, R)}\, d\xi}{P(\xi) - \lambda - i0}. \tag{5}$$

Lemma 2. Let the direction $\omega = (\omega_1, \dots, \omega_n)$, $|\omega| = 1$, be such that the surface $P(\xi) = \lambda$ is convex at the point P_ω, at which the outward normal to the surface is parallel to the vector ω and coincides with it in direction,

moreover, such a point P_ω is unique. Then for $R = |R|\omega$, as $|R| \to \infty$, the asymptotic representation holds

$$H_0(R; \lambda) = c(\omega) e^{-i\mu(\omega)|R|} / |R|^{(n-1)/2} + o(1/|R|^{(n-1)/2}). \tag{6}$$

Here $c(\omega)$ is a coefficient depending only on the gradient and the Gaussian curvature of the surface $P(\xi) = \lambda$ at the point P_ω; $\mu(\omega)$ is the projection of the point P_ω onto the direction ω.

A direction ω satisfying the conditions of Lemma 2 will be called **admissible**.

Lemmas 1 and 2 can be obtained by the method of the work [1].

2. Let now c_R be a real finite sequence ($c_R \neq 0$ if and only if $R \in \mathfrak{A}_1$, $\mathfrak{A}_1$ being a finite set). The operator $L = L_0 + C$, where C is the operator of multiplication by c_R, is a bounded self-adjoint operator. For $z = \lambda + i\varepsilon$, $\varepsilon > 0$, L has a resolvent, whose kernel we denote by $H(R, R_0; \lambda + i\varepsilon)$.

Theorem 1. *Let $\lambda \in (\alpha, \beta)$, and let the surface $P(\xi) = \lambda$ have no singular points. If the trigonometric polynomial $P(\xi) - \lambda$ is irreducible, then there exists the limit of the resolvent kernel $H(R, R_0; \lambda + i\varepsilon)$ as $\varepsilon \to 0$:*

$$\lim_{\varepsilon \to} H(R, R_0; \lambda + i\varepsilon) = H(R, R_0; \lambda). \tag{7}$$

Proof. Represent $H(R, R_0; \lambda + i\varepsilon)$ in the form

$$H_0(R, R_0; \lambda + i\varepsilon) = \sum_{R'} H_0(R - R'; \lambda + i\varepsilon) g_{R'}(R_0; \lambda + i\varepsilon), \tag{8}$$

where $g_R(R_0; \lambda+i\varepsilon)$ is an unknown sequence. $g_R(R_0; \lambda+i\varepsilon)$ satisfies the equation

$$g_R(R_0; \lambda + i\varepsilon) + c_R \sum_{R'} H_0(R - R'; \lambda + i\varepsilon) g_{R'}(R_0; \lambda + i\varepsilon) = \delta_{R_0}. \tag{9}$$

Since $c_R = 0$ for $R \notin \mathfrak{A}_1$, (9) is an algebraic system of equations. By Lemma 1, to prove Theorem 1 it is enough to establish the following lemma:

Lemma 3. *Under the assumptions of Theorem 1, the homogeneous equation*

$$g_R + c_R \sum_{R'} H_0(R - R'; \lambda) g_{R'} = 0 \tag{10}$$

has only the trivial solution.

Proof. Suppose there is a solution g_R of equation (10). Write (10) for $R \in \mathfrak{A}_1$ in the form

$$\sum_{R'} H_0(R - R'; \lambda) g_{R'} = -(1/c_R) g_R,$$

multiply the left- and right-hand sides by $\overline{g_R}$, and sum over all $R \in \mathfrak{A}_1$. Using (5), we obtain

$$\frac{1}{(2\pi)^n} \operatorname{v.p.} \int_\Omega \frac{|\tilde{g}(\xi)|^2 \, d\xi}{P(\xi) - \lambda} - \frac{i\pi}{(2\pi)^n} \int_{P(\xi)=\lambda} \frac{|\tilde{g}(\xi')|^2 \, d\xi'}{|\operatorname{grad} P(\xi')|} = -\sum_{R\in\mathfrak{A}_1} \frac{1}{c_R} |g_R|^2, \tag{11}$$

where

$$\tilde{g}(\xi) = \sum_{R\in\mathfrak{A}} {}^{1} g_R e^{i(\xi,R)}.$$

Taking the imaginary part in (11), we obtain

$$\int_{P(\xi)=\lambda} \frac{|\tilde{g}(\xi)|^2}{|\operatorname{grad} P(\xi')|} \, d\xi' = 0.$$

Consequently, $\tilde{g}(\xi) = 0$ when $P(\xi) - \lambda = 0$. Since $P(\xi) - \lambda$ is irreducible, the polynomial $\tilde{g}(\xi)$ is divisible by $P(\xi) - \lambda$ (see, for example, [2]), i.e.

$$\tilde{g}(\xi) = [P(\xi) - \lambda] \tilde{g}_1(\xi), \tag{12}$$

where $g_1(\xi)$ is again a trigonometric polynomial

$$\tilde{g}_1(\xi) = \sum_{R \in \mathfrak{A}_2} g_{1R} e^{i(\xi,R)},$$

$\mathfrak{A}_2$ is a finite set. From (12) it follows that g_{1R} is a finite sequence—

ity satisfying the equation

$$\sum_{R' \in \mathfrak{A}} a_{R'} g_{1,R-R'} - \lambda g_{1R} + c_R g_{1R} = 0. \tag{13}$$

By virtue of the conditions imposed on the set $\mathfrak{A}$, from this we obtain $g_{1R} \equiv 0$. Consequently, $g_R \equiv 0$. The lemma is proved.

3. Let us now consider the equation

$$\sum_{R' \in \mathfrak{A}} a_{R'} u_{R-R'} + c_R u_R - \lambda u_R = f_R, \tag{14}$$

where the right-hand side $f = \{f_R\}$ is finite.

Under the assumptions of Theorem 1, we apply to equation (14) the limiting absorption principle: the unique solution is singled out as the limit, as $\varepsilon \to 0$, of the solution of equation (14) with $\lambda + i\varepsilon$ instead of λ; moreover, this sought solution is given by the formula $u_R = \sum_{R'} H(R, R'; \lambda) f_{R'}$.

The unique solution of equation (14) can also be singled out by conditions at infinity of Sommerfeld type, but under more stringent assumptions.

Theorem 2. *Suppose that the conditions of Theorem 1 are satisfied and, moreover, the surface $P(\xi) - \lambda = 0$ is convex, consists of one closed component, and has Gaussian curvature everywhere different from zero, i.e. all ω are admissible. Suppose also that $a_R = a_{-R}$.*

Then for any finite sequence f_R there exists a unique solution of equation (14) satisfying at infinity the conditions

$$u_{R+S} - e^{-i\mu(\omega)(\omega,S)} u_R = o(1/|R|^{(n-1)/2}), \quad u_R = O(1/|R|^{(n-1)/2}), \tag{15}$$

where $R = |R|\omega$, and S is an arbitrary integer vector in R^n.

For general differential equations with constant coefficients, Sommerfeld conditions were obtained in (^{1},^{3}). In (^{4}), for equations of general form with variable coefficients, radiation conditions are formulated and the limiting absorption principle is justified.

4. The direct scattering problem consists in finding, for the difference equation

$$\sum_{R' \in \mathfrak{A}} a_{R'} u_{R-R'} + c_{Ru} R - \lambda u_R = 0 \tag{16}$$

the unique solution of the form $u_R = e^{i(k,R)} + v_R(k)$, where $k = (k_1, \dots, k_n)$ satisfies the equation $P(k) = \lambda$, $\lambda \in (\alpha, \beta)$. The function $e^{i(k,R)}$ is called a **plane wave** incident in the direction k, and $v_R(k)$ is the **scattered wave**.

Such a formulation of the problem is analogous to the formulation of the scattering problem for the Schrödinger equation ([5]) (see also ([6])). For difference equations the direct scattering problem was studied in ([7]).

For $v_R(k)$ one obtains the equation

$$\sum_{R' \in \mathfrak{A}} a_{R'} v_{R-R'} + c_{Rv} R - \lambda v_R = -c^{i(k,R)}_{Re}. \tag{17}$$

As was indicated in Sec. 3, its unique solution can be singled out either by the limiting absorption principle, using Theorem 1, or by conditions at infinity, using Theorem 2.

The scattered wave $v_R(k)$ has the following asymptotic behavior: for admissible ω and large $|R|$,

$$v_R(k) = A(\omega, k) c(\omega) e^{-i\mu(\omega)|R|} / |R|^{(n-1)/2} + o(1/|R|^{(n-1)/2}), \tag{18}$$

where $c(\omega)$ and $\mu(\omega)$ are the same as in Lemma 2; the coefficient $A(\omega, k)$, called the **scattering amplitude**, depends on the perturbation c_R.

5. The inverse scattering problem consists in recovering the perturbation c_R from the scattering amplitude $A(\omega, k)$ of the scattered wave $v_R(k)$. We shall assume that the region outside which the perturbation c_R vanishes is known: $c_R = 0$ for $|R| > M$.

Theorem 3. *For the unique recovery of the perturbation* c_R ($|R| \leqslant M$), *it is sufficient to specify the scattering amplitude* $A(\omega, k)$ *for all* $\omega = (\omega_1, \dots, \omega_n)$, $|\omega| = 1$, *from the neighborhood* $|\omega - \omega_0| < \delta$, *and all* $k = (k_1, \dots, k_n)$ *from the neighborhood* $|k - k_0| < \delta$ ($\delta > 0$ *small), where* ω_0 *and* k_0 *satisfy the following conditions: 1) denote* $\lambda_0 = P(k_0)$; *let* k_0 *be such that the surface* $P(\xi) - \lambda_0 = 0$ *has no singular points and the polynomial* $P(\xi) - \lambda_0$ *is irreducible; 2) the direction* ω_0 ($|\omega_0| = 1$) *is admissible for the surface* $P(\xi) = \lambda_0$.

It is possible to indicate, in a certain sense, an effective procedure for recovering c_R from the indicated scattering data.

The proof of Theorem 3 consists of several stages.

a) For fixed k, from $A(\omega, k)$ one recovers $V_R(k)$ for $|R| > M$. Here one constructs a certain trigonometric polynomial in many complex variables, connected with $v_R(k)$ by its values on a piece of the surface $P(\xi) = \lambda$ ($\lambda = P(k)$). The Bezout theorem (8) is used essentially.

b) From the scattered wave $v_R(k)$ one recovers the resolvent $H(R, R_0; \lambda)$ for fixed λ, R, R_0 ($|R|, |R_0| > M, \lambda \in P(k), \lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$, where $\varepsilon > 0$ is small). This is possible, since $v_R(k)$ and the difference $H(R, R_0; \lambda) - H_0(R, R_0; \lambda)$ satisfy equations differing only in their right-hand sides.

c) Using the analyticity of $H(R, R_0; z)$ in z in the upper half-plane and the Stieltjes inversion formula, we recover $(E_\lambda \delta_R, \delta_{R'})$ for all λ, R, R' ($|R|, |R'| > M$), where E_λ is the resolution of the identity of the operator L.

d) From $(E_\lambda \delta_R, \delta_{R'})$ for $|R|, |R'| > M$ one recovers the perturbation c_R for $|R| \leqslant M$. Here the spectral theory of difference operators, constructed in (9) (see also (10)), is used.

For the Schrödinger equation the inverse problem was considered in (11). The formulation of the inverse problem given above and the stages of the proof of Theorem 3 are analogous to the scheme of that work. In (12) the inverse scattering problem for a one-dimensional difference equation was considered.

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Institute of Mathematics
Academy of Sciences of the Ukrainian SSR

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Note: Figure translations are in progress. See original paper for figures.