ON DIFFERENTIATION AND QUASILINEAR OPERATORS IN THE SPACE OF FUNCTIONS WHOSE GENERALIZED DERIVATIVES ARE MEASURES

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Abstract

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ON DIFFERENTIATION AND QUASILINEAR OPERATORS IN THE SPACE OF FUNCTIONS WHOSE GENERALIZED DERIVATIVES ARE MEASURES

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In attempting to define a quasilinear operator $T(u)$ (see (4)) on discontinuous functions, difficulties of an analytical nature arise. For example, if one is concerned with the agreement between the divergent and nondivergent forms of writing, then in order to pass from the first to the second it is necessary to carry out generalized differentiation of discontinuous functions, which, as simple examples show (Sec. 2), does not obey the usual rules unless the corresponding definitions are given. The question of generalized differentiation of discontinuous functions is, of course, also of independent interest. Wishing to obtain formulas analogous to the usual ones, one has to introduce a new notion of superposition (Sec. 1). It is then used in defining the operator $T(u)$ (Sec. 3). For continuous functions the superposition defined in Sec. 1 coincides with the usual one. The considerations are carried out in the space of functions whose generalized derivatives are locally measures. Such spaces have been studied by a number of authors (see ([1,2]) and the references therein).

1. Let Ω be a domain of the n-dimensional Euclidean space R^n; B the collection of all bounded Borel subsets whose closure belongs to Ω; $A(\Omega)$ the space of real countably additive set functions, defined and finite on B; $V(\Omega)$ the space of functions, defined and locally summable in Ω, whose first generalized derivatives belong to $A(\Omega)$. If $u(x)$ is a function defined in Ω, $\mu \in A(\Omega)$, and the function u is summable with respect to μ on each set $E \in B$, then multiplication of u by μ is understood in the sense

$$(u\mu)(E) = \int_E u(x)\mu(dx) \qquad (E \in B), \tag{1}$$

so that $u\mu \in A(\Omega)$.

Let a be a unit vector in R^n; $x_0 \in \Omega$; $u(x)$, $v(x)$ functions defined in Ω; $w \in V(\Omega)$. Obviously, if the ball $|x - x_0| \le r$ belongs to Ω, then the generalized derivative $\partial w/\partial a$ in the direction a is a measure finite on the Borel subsets of this ball. If there exists a ball $|x - x_0| \le r$ belonging to Ω such that the set $\{x : u(x) \ne v(x),\ |x - x_0| \le r\}$ has $\partial w/\partial a$-measure 0, then we shall write $u \sim_a v(w; x_0)$. The following notation is used: 1) $u \sim_a v(w)$, if $u \sim_a v(w; x_0)$ for all $x_0 \in \Omega$; 2) $u \sim_a v$, if $u \sim_a v(w)$ for all $w \in V(\Omega)$; 3) $u \approx v$, if $u \sim_a v$ for all unit vectors a; 4) $u \sim v$, if u is equivalent to v with respect to the ordinary Lebesgue measure; 5) $E_1 \approx E_2$, if $\chi_{E_1} \approx \chi_{E_2}$, where $\chi_{E_t}(x)$ is the restriction of the characteristic function of the set E_t to Ω $(t = 1, 2)$.

Theorem 1. *If $u \in V(\Omega)$, then there exists a pair of functions u^*_a, u_{*a}, defined and finite for all $x \in \Omega$, such that $u^*_a \sim_a \bar{u}_a$, $u_{*a} \sim_a \underline{u}_a$ for all unit vectors a, where*

$$\bar{u}_a(x) = \overline{\lim_{h\to 0}} \int_0^1 u(x + tha)\,dt, \qquad \underline{u}_a(x) = \underline{\lim_{h\to 0}} \int_0^1 u(x + tha)\,dt \tag{2}$$

(on the set of points x, where there is no integral in (2) for sufficiently small h, one may assign $\bar{u}_a(x)$, $\underline{u}_a(x)$ arbitrarily).

Obviously, u^* and u_* are determined by u uniquely up to equivalence $\asymp$. It is easy to see that if $v \sim u$, then $v^* \asymp u^*$, $v_* \asymp u_*$.

The proof of the theorem is based on the following lemma.

Lemma. Let $u \in V(\Omega)$, and let e_i be the unit vector in the direction of the axis x_i $(i = 1, \dots, n)$. Then, for any $i, j = 1, \dots, n$, the set of those points $x \in \Omega$ for which $\bar{u}_{e_i}(x) \ne \bar{u}_{e_j}(x)$ can be represented in the form $A_{ij} \cup B_{ij}$, where A_{ij} (B_{ij}) has projection onto the plane $x_i = 0$ $(x_j = 0)$ whose $(n-1)$-dimensional Lebesgue measure is equal to 0.

The construction of the function $u^*(x)$ is carried out as follows. Let

$$E_i = \bigcup_{k\ne i} A_{ik} \cup \bigcup_{k\ne i} B_{ki} \cup C_i, \qquad E_i' = \Omega \setminus E_i \quad (i = 1, \dots, n),$$

where C_i is the set of those $x \in \Omega$ for which $\bar{u}_{e_i}(x)$ does not take finite values. Then E_i has projection onto the plane $x_i = 0$ whose $(n-1)$-dimensional Lebesgue measure is equal to zero. For $x \in E_i'$ we put $u^*(x) = u_{e_i}(x)$ $(i = 1, \dots, n)$.

For $x \in \bigcap_{i=1}^n E_i$, $u^*(x)$ may be assigned arbitrarily. It is easy to see that the function $u^*(x)$ so defined is single-valued. It is proved that it satisfies the conditions of Theorem 1. To construct the function $u_*(x)$ it suffices to construct $v^*(x)$, where $v = -u$, and put $u_*(x) = v^*(x)$.

The generalization to the case when u is a vector is:

$$u^* \overset{a}{\sim} \bar{u}_a, \qquad u_* \overset{a}{\sim} \underline{u}_a,$$

where

$$\bar{u}_a(x) = \max\{u_a(x), u_{-a}(x)\}, \qquad \underline{u}_a(x) = \min\{u_a(x), u_{-a}(x)\}$$

(in the sense of lexicographic ordering),

$$u_a(x) = \lim_{h\to+0} \int_0^1 u(x+tha)\,dt.$$

Let $u = (u_1, \dots, u_p) \in V(\Omega)$; let $f(x,u)$ be a function whose domain of definition contains the set of points of R^{n+p} with coordinates

$$(x, u^*(x)\alpha + u_*(x)(1-\alpha)),$$

where $0 \le \alpha \le 1$, $x \in E \subseteq \Omega$, $E \asymp \Omega$, such that there exists the integral

$$\bar{f}(x,u(x)) = \int_0^1 f(x, u^*(x)\alpha + u_*(x)(1-\alpha))\,d\alpha \qquad (x \in E).$$

By definition, $\bar{f}(x,u(x))$ $(x \in E)$ is the **functional superposition** corresponding to the pointwise superposition $f(x,u(x))$. For $x \in \Omega \setminus E$, $\bar{f}(x,u(x))$ may be extended arbitrarily.

Taking into account the possible arbitrariness in the choice of E, u^*, and u_*, it is easy to see that $\bar{f}(x,u(x))$ is determined uniquely up to equivalence $\asymp$. Moreover, since $u^* \sim u_* \sim u$, it follows that $\bar{f}(x,u(x)) \sim f(x,u(x))$; however, the equivalence $\sim$ is too coarse for the purposes of generalized differentiation.

Remark 1. In the definition given, what is in fact used is not the membership $u \in V(\Omega)$, but the existence of u_* and u^*; therefore this definition makes sense for a broader class of functions.

Remark 2. For $f(u) = u$ one obtains $\bar{u} = \frac{1}{2}(u^* + u_*)$, which may be taken as a normalization up to $\asymp$-equivalence in each class of $\sim$-equivalence.

2. Theorem 2. Suppose: 1) $u(x) = (u_1(x), \dots, u_p(x)) \in V(\Omega)$; 2) $f(u)$ is defined, continuous, and has continuous first derivatives in a domain $G \subset R^p$ containing the points $u^*(x)\alpha + u_*(x)(1-\alpha)$ for all $x \in \Omega$, $0 \le \alpha \le 1$; 3) for each $i = 1, \dots, n$, $k = 1, \dots, p$ there exists a function $\Phi_{ik}(x)$, locally summable in Ω with respect to the measure $D_i u_k$, such that

$$\left|\bar{f}_{u_k}(\bar{u}(x))\right| \le \Phi_{ik}(x) \qquad (x \in \Omega),$$

where $D_i u_k$ is the generalized derivative $\partial u_k / \partial x_i$, and $f_{u_k} = \partial f / \partial u_k$.

Then $f(u(x)) \in V(\Omega)$, the multiplication of $\bar{f}_{u_k}(u(x))$ by $D_i u_k$ is defined in the sense of (1), and the formula holds

$$D_i f(u(x)) = \sum_{k=1}^{p} \bar{f}_{u_k}(u(x)) D_i u_k \quad (i = 1, \dots, n). \tag{3}$$

Remark 1. Condition 3) is satisfied automatically if u_k $(k = 1, \dots, p)$ are locally bounded in Ω.

Remark 2. The conditions 2) on $f(u)$ can be weakened by means of a limiting passage.

Remark 3. Formula (3), generally speaking, is false if one uses not functional but pointwise superposition. For example, let $u(x)$ be the characteristic function of some set, so that for every $m > 0$, $u^m(x) = u(x)$. If the formula $D_i u^m = m u^{m-1} D_i u$ were true, then we would have

$$0 = D_i(u^4 - u) = 4u^3 D_i u - D_i u = 4u D_i u - D_i u = 2D_i u^2 - D_i u = 2D_i u - D_i u = D_i u.$$

If, in particular, in (3) we put $u = (u_1, u_2)$, $f(u) = u_1 u_2$, then the formula for differentiating a product is obtained

$$D_i(u_1 u_2) = \bar{u}_1 D_i u_2 + \bar{u}_2 D_i u_1 \quad (i = 1, \dots, n).$$

3. Let an equation be given

$$T(u) \equiv \sum_{k=1}^{n} a_k(x, u) D_k u + a_0(x, u) = 0, \tag{4}$$

where a_k are matrices of order p; u is a vector of p elements. The question arises in what sense the operator $T(u)$ can be defined on discontinuous functions. Here the case is considered when $u \in V(\Omega)$.

Suppose first that (4) is obtained by formally expanding the differentiation in the equation

$$\sum_{k=1}^{n} D_k b_k(x, u) + b_0(x, u) = 0, \tag{5}$$

where b_k are vector-functions. The definition of a generalized solution of this equation is known. It follows from Theorem 2 that if the conditions formulated in it are satisfied, then equation (4) is equivalent to equation (5), and the superposition $a_k(x, u(x))$ must be understood as functional. However, in studying equation (4) it is inadvisable to restrict oneself only to functional superposition, since, for example, multiplication of the left- and right-hand sides of equation (4) by a certain matrix depending on u leads to an equation in which the superposition may no longer be functional. (Therefore the resulting equation, generally speaking, can no longer be written in the form (5) for discontinuous u, even if this is possible for smooth ones. This, in particular, explains the nonequivalence, on discontinuous solutions, of two different divergent forms of writing certain concrete equations; an example of such nonequivalence is given in ([3]), p. 92.)

Suppose now that (4) is considered independently of (5). From the preceding it is clear how the natural domain of definition of the operator $T(u)$ on vector-functions from $V(\Omega)$ should be understood. At the same time it should only be emphasized that the following assumptions are meant: the form of the superposition $a_k(x, u(x))$ is prescribed (generally speaking, as a combination of functional and pointwise superposition), and multiplication is understood in the sense of (1) (strictly speaking, instead of $a_0(x, u(x))$ one should write $a_0(x, u(x))\mu$, where μ is Lebesgue measure).

Everything said is also valid for quasilinear differential operators of order $m > 1$, if instead of the space $V(\Omega)$ one considers the space $V^{(m)}(\Omega)$ of functions all of whose generalized derivatives up to order $m-1$ belong to $V(\Omega)$.

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Note: Figure translations are in progress. See original paper for figures.