THE INVERSE PROBLEM FOR THE DIRAC SYSTEM

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M. G. GASYMOV, B. M. LEVITAN

THE INVERSE PROBLEM FOR THE DIRAC SYSTEM

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§ 1. Consider the system of Dirac differential equations

$$\{B\,d/dx + Q(x)\}y = \lambda y, \qquad 0 \le x < \infty, \tag{1}$$

where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Q(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & r(x) \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Here we assume that p, q, and r are real functions, integrable on every finite interval in $[0, \infty)$. Denote by $\varphi(x, \lambda) = \begin{pmatrix} \varphi_1(x, \lambda) \\ \varphi_2(x, \lambda) \end{pmatrix}$ the solution of equation (1) with the initial conditions:

$$\varphi_1(0, \lambda) = \sin\alpha, \qquad \varphi_2(0, \lambda) = -\cos\alpha \quad (\alpha \text{ is a real number}). \tag{2}$$

Let $f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} \in L_2(0, \infty)$. Put

$$F_n(\lambda) = \int_0^n \{f_1(x)\varphi_1(x, \lambda) + f_2(x)\varphi_2(x, \lambda)\}\,dx.$$

From the general theory of differential operators it is known that for every matrix $Q(x)$ and every number α there exists a unique nondecreasing function $\rho(\lambda)$ (in what follows we call it the spectral function of problem (1)—(2)), defined on the entire real axis, such that

$$\int_0^\infty \{f_1^2(x) + f_2^2(x)\}\,dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} F_n^2(\lambda)\,d\rho(\lambda).$$

In this paper we indicate necessary and sufficient conditions for a nondecreasing function $\rho(\lambda)$ to be the spectral function of an equation of type (1). The analogous problem for the Sturm–Liouville equation was solved in (^{1}) (see also (^{2})). Several papers ($^{3-5}$) have also been devoted to the inverse problem for the Dirac system; however, the results of these papers are mainly of a conditional nature, since in them only a procedure is indicated for reconstructing the Dirac system from the function $\rho(\lambda)$, and the properties of this function are not investigated. The first question that naturally arises here is the question of the unique determination of an equation of type (1) by its spectral function.

It is not difficult to see that the answer to this question is negative. Indeed, if $\omega(x)$ is an absolutely continuous function and

$$A(x) = \begin{pmatrix} \cos\omega(x) & \sin\omega(x) \\ -\sin\omega(x) & \cos\omega(x) \end{pmatrix},$$

then $\psi(x,\lambda) = A(x)\varphi(x,\lambda)$ satisfies the self-adjoint equation

$$B\psi' + \{A^{-1}BA' + A^{-1}QA\}\psi = \lambda\psi$$

and, as is not difficult to verify, $\psi(x,\lambda)$ generates Parseval' s equality with the same measure $\rho(\lambda)$. The indicated remark shows that an equation of type (1) is not uniquely determined by its spectral function. Therefore it is of interest to find such a form (canonical form) for equation (1) under which it is uniquely determined

with respect to $\rho(\lambda)$. For this, we shall try to choose the matrix $A(x)$ so that equation (1) and the initial conditions (2) for $\psi(x,\lambda)$ are as simple as possible, and we shall require that

$$\operatorname{Sp}\{A^{-1}BA' + A^{-1}QA\} = 0 \quad \text{and} \quad \psi_1(0,\lambda) = 0, \quad \psi_2(0,\lambda) = -1.$$

The resulting system has the unique solution

$$\omega(x) = \frac{1}{2}\int_0^x (p+r)\,dx + \alpha.$$

The preceding arguments show that any equation of the form (1), by means of an orthogonal transformation, is reduced to the canonical form $By' + Q_1(x)y = \lambda y$, where $\operatorname{Sp} Q_1(x) = 0$, and condition (2) is reduced to the form $y_1(0) = 0$, $y_2(0) = -1$. Therefore, everywhere below we shall consider problem (1)–(2) under the condition $r = -p$ and $\alpha = 0$. It turns out that in this form equation (1) is uniquely determined by $\rho(\lambda)$. The inverse problem consists in determining equation (1) (with $r = -p$ and $\alpha = 0$) from $\rho(\lambda)$. Before proceeding to the solution of this problem, we shall study some questions connected with the direct problem.

Theorem 1. If $\varphi(x, \lambda)$ is the solution of equation (1) with initial conditions $\varphi_1(0) = 0,\ \varphi_2(0) = -1$, then there exists a matrix function $K(x, t)$, summable on every finite interval, such that

$$\varphi(x, \lambda) = f(x, \lambda) + \int_0^x K(x, t) f(t, \lambda)\, dt, \tag{3}$$

where

$$f(x, \lambda) = \begin{pmatrix} \sin \lambda x \\ -\cos \lambda x \end{pmatrix}.$$

Moreover, if $Q(x)$ is absolutely continuous, then $K(x, t)$ satisfies the differential equation

$$B \frac{\partial}{\partial x} K(x, t) + Q(x) K(x, t) = -\frac{\partial}{\partial t} K(x, t) B \tag{4}$$

and the initial conditions

$$BK(x, x) - K(x, x)B = -Q(x), \tag{5}$$

$$K_{21}(x, 0) = K_{11}(x, 0) = 0. \tag{6}$$

Conversely, if $K(x, t)$ satisfies equation (4) and conditions (5)–(6), then $\varphi(x, \lambda)$, defined by formula (3), satisfies equation (1) and the initial conditions $\varphi_1(0, \lambda) = 0,\ \varphi_2(0, \lambda) = -1$.

The existence of the kernel $K(x, t)$ of the transformation operator (3) makes it possible to solve the inverse problem posed above by the method of I. M. Gel'fand and B. M. Levitan (see (^{1},^{2})). For this we shall need some properties of the function $\rho(\lambda)$.

Theorem 2. If $\rho(\lambda)$ is the spectral function of problem (1)–(2), then it has the following properties:

1. $\rho(\lambda)$ increases monotonically on the whole axis.
2. Let

$$G(\lambda) = \int_0^\infty \{g_1 \sin \lambda x - g_2 \cos \lambda x\}\, dx,$$

where g_1 and g_2 are finite functions from $L_2(0, \infty)$. Then, if

$$\int_{-\infty}^{\infty} G^2(\lambda)\, d\rho(\lambda) = 0, \tag{7}$$

then $G(\lambda) \equiv 0$ and, consequently, $g_1(x) = g_2(x) \equiv 0$.

3. *The limit*

$$\lim_{N\to\infty} \int_{-N}^{N} \begin{pmatrix} \sin\lambda x \\ -\cos\lambda x \end{pmatrix} (\sin\lambda y,\ -\cos\lambda y)\, d\left[\rho(\lambda) - \frac{1}{\pi}\lambda\right] = F(x,y) \tag{8}$$

exists boundedly in any finite domain of variation of x, y. Moreover, $F(x,y)$ has, with respect to both variables, as many derivatives as $Q(x)$ has.

Now, repeating almost word for word the reasoning from § 2 of Chapter I of paper (2), one can prove that the kernel $K(x,y)$ of the operator (3) satisfies the integral equation

$$F(x,y) + K(x,y) + \int_0^x K(x,s)F(s,y)\, ds = 0, \qquad 0 \leqslant y < x. \tag{9}$$

This equation enables us to solve completely the inverse problem for the Dirac system. The following theorem holds:

Theorem 3. *In order that the function $\rho(\lambda)$ be the spectral function of some equation of the form (1) with matrix*

$$Q(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix},$$

having n locally summable derivatives, and with initial conditions $y_1(0) = 0$, $y_2(0) = -1$, it is necessary and sufficient that the function $\rho(\lambda)$ possess all the properties from Theorem 2.

We outline the proof of this theorem. Construct the function $F(x,y)$ by formula (8) and consider the integral equation (9). We shall show that it has a unique solution. For this it is enough to prove that, for each x, the homogeneous equation

$$(g_1(y),\, g_2(y)) + \int_0^x (g_1(t),\, g_2(t))F(t,y)\, dt = 0$$

has only the zero solution. Let $(g_1(y),\, g_2(y))$ be a solution of this equation. Then

$$\int_0^x \{g_1^2(y) + g_2^2(y)\}\, dy + \int_0^x \int_0^x (g_1(t),\, g_2(t))F(t,y) \begin{pmatrix} g_1(y) \\ g_2(y) \end{pmatrix} dt\, dy = 0,$$

or

$$\int_{-\infty}^{\infty} G^2(\lambda)\, d\rho(\lambda) = 0,$$

where

$$G(\lambda) = \int_0^x \{g_1(t)\sin\lambda t - g_2(t)\cos\lambda t\}\, dt.$$

By property 2 of the function $\rho(\lambda)$, $G(\lambda) \equiv 0$, i.e. $g_1 \equiv g_2 \equiv 0$. Consequently, equation (9) has a unique solution. Moreover, $K(x, y)$ has as many derivatives as $F(x, y)$ has. Now, with the aid of the kernel $K(x, y)$ found, construct the function $\varphi(x, \lambda)$ by formula (3). Then it is not difficult to prove that $\varphi(x, \lambda)$ satisfies the equation $B\varphi' + Q(x)\varphi = \lambda\varphi$, where $Q(x) = K(x, x)B - BK(x, x)$, and the conditions $\varphi_1(0, \lambda) = 0$, $\varphi_2(0, \lambda) = -1$. It is easy to verify that $\operatorname{Sp} Q(x) = 0$ and $Q^*(x) = Q(x)$. Moreover, if $F(x, y)$ has n derivatives, then $Q(x)$ has the same number of derivatives. It remains to show that $\varphi(x, \lambda)$ generates Parseval' s equality with respect to $\rho(\lambda)$. This is done approximately in the same way as in § 5 of Chapter I of paper (2).

§ 2. We now briefly dwell on the inverse problem for the Dirac system whose coefficients have a nonintegrable singularity at zero. Let $\varphi(x) = \binom{\varphi_1(x)}{\varphi_2(x)}$ be the solution of equation (1) with the initial conditions $\varphi_1(0) = 0$, $\varphi_2(0) = -1$ for $\lambda = 0$. Put

$$A(x, t) = \left[\int_0^x \{\varphi_1^2(t) + \varphi_2^2(t)\}\, dt\right]^{-1} \begin{pmatrix} \varphi_1(x)\varphi_1(t) & \varphi_1(x)\varphi_2(t) \\ \varphi_1(t)\varphi_2(x) & \varphi_2(x)\varphi_2(t) \end{pmatrix},$$

$$\Phi(x, \lambda) = \varphi(x, \lambda) - \int_0^x A(x, t)\varphi(t, \lambda)\, dt.$$

It is not difficult to verify that $\Phi(x, \lambda)$ is a solution of the equation

$$B\Phi' + Q(x)\Phi -$$

$$- \left[\int_0^x \{\varphi_1^2(t) + \varphi_2^2(t)\}\, dt\right]^{-1} \begin{pmatrix} 2\varphi_1(x)\varphi_2(x) & \varphi_1^2(x) - \varphi_2^2(x) \\ \varphi_1^2(x) - \varphi_2^2(x) & -2\varphi_1(x)\varphi_2(x) \end{pmatrix} \Phi = \lambda\Phi \quad (10)$$

and, as $x \to 0$, $\dfrac{1}{\lambda x}\Phi(x, \lambda) \to \binom{1}{0}$. It is obvious that this equation has a singularity of the form $\dfrac{1}{x}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Therefore any other solution of equation (10), linearly independent of $\Phi(x,\lambda)$, tends to infinity as $x=0$. A direct calculation proves that $\Phi(x,\lambda)$ generates Parseval' s equality to the same extent as does $\varphi(x,\lambda)$. By analogous arguments one can pass from a singularity of the type $\frac{1}{x}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ to a singularity of the type $\frac{2}{x}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, etc. Thus we arrive at the following result:

Theorem 4. *If $\rho(\lambda)$ is the spectral function of an equation of type (1) without singularities, then it is also the spectral function of equations of the type*

$$By' + Q_l(x)y - \frac{l}{x}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} y = \lambda y,$$

where $Q_l(x)$ is a function summable on any finite interval of $[0,\infty)$, and $\operatorname{Sp} Q_l(x) = 0,\ Q_l^*(x) = Q_l(x),\ l = 0, \pm 1, \pm 2, \ldots$.

Moscow State University
named after M. V. Lomonosov

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Note: Figure translations are in progress. See original paper for figures.

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