AN INVERSE PROBLEM FOR A NON-SELF-ADJOINT OPERATOR

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Abstract

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MATHEMATICS

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AN INVERSE PROBLEM FOR A NON-SELF-ADJOINT OPERATOR

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In the present article we consider the inverse problem of spectral analysis for a non-self-adjoint operator, analogous to the inverse problem of scattering theory (see ([1])). As we shall see below, in the non-self-adjoint case the inverse problem can be posed and solved by a method very close to that developed in ([1]). The changes to which we were forced to resort are connected with the fact that the "scattering function" of a non-self-adjoint operator is not only not unitary but, generally speaking, is not summable, since it may have singularities of pole type (the case of an operator with spectral singularities). In addition, in the non-self-adjoint case it is possible for not only eigenfunctions but also associated functions to exist. True, the latter of these circumstances (which is of important fundamental significance) only slightly complicates the problem.

Consider the operator L, generated in the Hilbert space $L^2(0,\infty)$ by the differential expression $ly = -y'' + p(x)y$ ("potential," $p(x)$ a complex-valued function) and the boundary condition $y(0) = 0$. For simplicity we impose on the potential $p(x)$ the following condition, introduced by M. A. Naimark in ([2]): there exists an $\varepsilon > 0$ such that

$$\int_0^\infty e^{\varepsilon x} |p(x)|\, dx < \infty. \tag{1}$$

Accordingly, the problem of reconstructing the potential $p(x)$ from the "scattering data" (see below) will be solved by us in the class of functions satisfying condition (1).

Let

$$e(x,\rho) = e^{ix\rho} + \int_x^\infty K(x,t) e^{it\rho}\, dt, \qquad \operatorname{Im}\rho \geq 0,\ x \geq 0, \tag{2}$$

be a special solution of the equation $ly = \rho^2 y$ (see ([1])). Put

$$e(\rho) = e(0, \rho), \qquad s(\rho) = e(-\rho)/e(\rho); \tag{3}$$

the functions $e(\rho)$ and $s(\rho)$ will be called, respectively, the denominator (of the kernel of the resolvent of the operator L) and the scattering function (of the operator L).

We introduce the following definition. A function $E(\rho)$ will be called a function of type (E) in the half-plane $\operatorname{Im}\rho > -\varepsilon_0$ (ε_0 is some positive number) if:

1) $E(\rho)$ is holomorphic for $\operatorname{Im}\rho > -\varepsilon_0$ and, for every $\eta < \varepsilon_0$,

$$E(\rho) = 1 + O(1/\rho), \qquad |\rho| \to \infty$$

uniformly in the half-plane $\operatorname{Im}\rho \geq -\eta$;

2) $E(\rho) \neq 0$ for $0 < |\operatorname{Im}\rho| < \varepsilon_0$;
3) if $\rho \neq 0$, $\operatorname{Im}\rho = 0$ and $E(\rho) = 0$, then $E(-\rho) \neq 0$;
4) if $E(0) = 0$, then $E'(0) \neq 0$.

By virtue of condition (1), the denominator $e(\rho)$ is a function of type (E) in the half-plane $\operatorname{Im}\rho > -\varepsilon_0$ for some positive $\varepsilon_0 < \varepsilon/2$. Therefore $e(\rho)$ has only a finite number of zeros in the domain $\operatorname{Im}\rho \geq 0$, $|\rho| \neq 0$. We shall call these zeros the **singular numbers** (of the operator L). We denote the nonreal singular numbers by $\rho_1, \dots, \rho_\alpha$, and the real ones by $\rho_{\alpha+1}, \dots, \rho_\beta$. The multiplicity m_k of the root ρ_k of the equation $e(\rho) = 0$ will be called the **multiplicity of the singular number** ρ_k, $k = 1, \dots, \beta$.

As is known, the numbers $\lambda_k = \rho_k^2$, $k = 1, \dots, \alpha$, form the point spectrum of the operator L, while the numbers $\lambda_k = \rho_k$, $k = \alpha + 1, \dots, \beta$ (the so-called spectral singularities) belong to the continuous spectrum (which fills the entire half-axis $\lambda \geq 0$), but play a certain special role (see ([2],[3])).

Let us introduce one more definition. We shall say that $S(\rho)$ is a **function of type** (S) in the strip $|\operatorname{Im}\rho| < \varepsilon_0$, if:

1) $S(\rho)$ is meromorphic in the strip $|\operatorname{Im}\rho| < \varepsilon_0$ and, for every $\eta < \varepsilon_0$,

   $$S(\rho) = 1 + O(1/\rho), \qquad |\rho| \to \infty,$$

   uniformly in the strip $|\operatorname{Im}\rho| \leq \eta$;

2) $S(\rho)$ has no nonreal poles in the strip $|\operatorname{Im}\rho| < \varepsilon_0$ and

   $$S(\rho)S(-\rho) = 1;$$

   in particular, $S(0) = \pm 1$.

It is easy to see that the scattering function $s(\rho)$ of an operator L satisfying condition (1) is a function of type (S) in the strip $|\operatorname{Im}\rho| < \varepsilon_0$ for some positive $\varepsilon_0 < \varepsilon/2$.

Let us now consider the problem of **symmetric factorization** of a function $S(\rho)$ of type (S) in the strip $|\operatorname{Im}\rho| \geq \varepsilon_0$. This problem consists in determining a function $E(\rho)$ of type (E) in the half-plane $\operatorname{Im}\rho > -\varepsilon_0$, satisfying the equation

$$E(-\rho)/E(\rho) = S(\rho), \tag{4}$$

having the prescribed numbers $R_1, \dots, R_\nu$ (from the half-plane $\operatorname{Im}\rho \geq \varepsilon_0$) as its roots of prescribed multiplicity respectively $M_1, \dots, M_\nu$; in addition, we require that $E(0) \neq 0$ in the case when $S(0) = 1$, and that $E(0) = 0$ in the case when $S(0) = -1$.

Let us introduce the concept of the **index** of the problem under consideration (we shall denote this index by $\operatorname{ind} S$). Let $\mathcal{L}$ be a curve lying in the strip $|\operatorname{Im}\rho| < \varepsilon_0$, going from $-\infty$ to $+\infty$, such that all zeros of $S(\rho)$ lie above $\mathcal{L}$, and all poles of $S(\rho)$ lie below $\mathcal{L}$. The increment divided by 2π of a continuous branch of $\arg S(\rho)$, as ρ traverses $\mathcal{L}$ from $-\infty$ to $+\infty$, is, by definition, $\operatorname{ind} S$.

Lemma. *For the solvability of the problem of symmetric factorization it is necessary and sufficient that*

$$\operatorname{ind} S + 2(M_1 + \dots + M_\nu) + \tfrac{1}{2}[1 - S(0)] = 0. \tag{5}$$

If this condition is fulfilled, then the solution of the problem is unique.

Corollary. *The denominator $e(\rho)$ is uniquely determined by specifying the scattering function $s(\rho)$, the nonreal singular numbers $\rho_1, \dots, \rho_\alpha$ and their multiplicities $m_1, \dots, m_\alpha$. The scattering function $s(\rho)$ and the multiplicities $m_1, \dots, m_\alpha$ satisfy a relation analogous to (5).*

Let $e_1(x, \rho)$ be the solution of the equation $ly = \rho^2 y$ which, for $\operatorname{Im}\rho > 0$, satisfies the condition $e_1(x, \rho)\exp(ix\rho) \to 1$ as $x \to \infty$. Put

$$F_k(x) = i \operatorname*{Res}_{|\rho=\rho_k} \frac{e_1(0, \rho)}{e(\rho)} e^{ix\rho}, \qquad M_k(x) = e^{-ix\rho_k} F_k(x). \tag{6}$$

The functions $M_1(x), \dots, M_\alpha(x)$ will be called the normalizing polynomials (of the operator L)* . Obviously, the degree of the polynomial $M_k(x)$ is equal to $m_k - 1$, where m_k is the multiplicity of the nonreal singular number ρ_k.

The scattering function $s(\rho)$, the nonreal singular numbers $\rho_1, \dots, \rho_\alpha$, and the normalizing polynomials $M_1(x), \dots, M_\alpha(x)$ will be called the scattering data (of the operator L). Just as in the self-adjoint case (see ([1])), the scattering data are determined by the asymptotics at infinity of the principal functions of the operator L, which generate the corresponding Parseval equality. From the preceding it is clear that specifying the scattering data is equivalent to specifying the denominator $e(\rho)$ and the normalizing polynomials $M_1(x), \dots, M_\alpha(x)$.

Theorem 1. *The scattering data uniquely determine the operator L.* The proof of this theorem for the nonself-adjoint operator differs in some details from the proof of the analogous theorem in the self-adjoint case. If real singular numbers $\rho_{\alpha+1}, \dots, \rho_\beta$ exist, then they are poles of the scattering function and, consequently, the function $s(\rho) - 1$ is not the Fourier transform (in the classical sense) of any summable function. In this connection we shall treat $s(\rho)$ as a generalized function. Since $s(\rho)$ has singularities of power type, we must choose a definite regularization of this function. For our purpose all regularizations are equally suitable, and for definiteness we shall introduce into consideration the (inverse) Fourier transform of the function $s(\rho) - 1$, setting

$$f_s^+(x) = \frac{1}{2\pi} \int_+ [s(\rho) - 1] e^{ix\rho} \, d\rho; \tag{7}$$

here the contour of integration in (7) is any straight line contained in the strip $|\operatorname{Im} \rho| < \varepsilon_0$**.

Theorem 2. *The kernel $K(x,t)$ (see formula (2)) satisfies the integral equation*

$$K(x,t) = \int_x^\infty K(x,u) f(u+t) \, du + f(x,t), \qquad 0 \le x \le t < \infty, \tag{8}$$

where

$$f(x) = f_s^+(x) - \{M_1(x) e^{ix\rho_1} + \dots + M_\alpha(x) e^{ix\rho_\alpha}\}, \qquad x \ge 0. \tag{9}$$

By analogy with the self-adjoint case (see ([1])), we shall call the integral equation (8) the basic equation, since it plays the principal role in reconstructing the potential $p(x)$ from the scattering data. Indeed, possessing the scattering data, with the aid of formulas (7) and (9) we can construct the function $f(x)$, and then from the basic equation (8) determine the kernel $K(x,t)$.

The potential $p(x)$ is expressed in terms of the kernel $K(x,t)$ by the formula

$$p(x) = -2 \frac{d}{dx} K(x,x). \tag{10}$$

We shall say that $F(x)$ is a function of type (F, ε) if:

1) for $x \ge 0$ there exists a continuous derivative $F'(x)$, and $F'(x) \exp(-\varepsilon t/2)$ is summable on the half-axis $x \ge 0$;

* If the operator L is self-adjoint, then the normalizing polynomials turn into normalizing multipliers (see ([1])).

** Such a choice of the contour of integration corresponds to the regularization which assigns to the rational fraction $(\rho - \rho_0)^{-m}$ the generalized function $(\rho - \rho_0 + i0)^{-m}$.

2) if $y_x(t)\exp(-\varepsilon t/2)$ is summable on the half-axis $t > x$ and

$$\eta_x(t) = \int_x^\infty y_x(u)F(u+t)\,du, \qquad t > x, \tag{11}$$

then $y_x(t) = 0$ for $t > x$.

It turns out that the kernel $f(x)$ of the basic equation (8) (corresponding to the operator L satisfying condition (1)) is a function of type (F, ε). From this fact and formula (10), in particular, Theorem 1 follows.

Let us note that, in the self-adjoint case, the proof that equation (11) has no nontrivial solutions for $F(x) = f(x)$ essentially uses the unitarity of the scattering function (see (1)). In the non-self-adjoint case this is no longer so; however, the author has succeeded in finding an unexpectedly simple proof of the absence of nontrivial solutions, suitable both in the self-adjoint and in the non-self-adjoint cases.

It turns out that after the substitution $y_x(t) = z_x(t) + \int_x^t K(\xi,t)z_x(\xi)\,d\xi$, $t \geq x$, equation (11) (for $F = f$) is transformed into a homogeneous Volterra equation with a rapidly decreasing kernel, whence the asserted uniqueness follows.

The necessary conditions described above are also sufficient for some collection of quantities to be the collection of scattering data of some operator. Namely, the following assertion is true.

Theorem 3. Let there be given some function $s(\rho)$ of type (S) in the half-plane $\operatorname{Im}\rho > -\varepsilon_0$, $\varepsilon_0 > 0$, some numbers $\rho_1, \dots, \rho_\alpha$ such that $\operatorname{Im}\rho_1 \geq \varepsilon_0, \dots, \operatorname{Im}\rho_\alpha \geq \varepsilon_0$, and some polynomials $M_1(x), \dots, M_\alpha(x)$ of degrees $m_1 - 1, \dots, m_\alpha - 1$. Suppose that these data are consistent in the sense that

$$\operatorname{ind} s + 2(m_1 + \dots + m_\alpha) + \frac{1}{2}[s(0) - 1] = 0. \tag{12}$$

Starting from them, construct the function $f(x)$ by means of formulas (7) and (9), and suppose that $f(x)$ is a function of type (F, ε) for some $\varepsilon > 2\varepsilon_0$. Then equation (8) has a unique solution $K(x,t)$, and the function $p(x)$, determined by formula (10), satisfies condition (1). The scattering data of the operator L corresponding to the potential (10) coincide with the initially prescribed quantities $s(\rho), \rho_1, \dots, \rho_\alpha, M_1(x), \dots, M_\alpha(x)$.

Let us note that the necessary and sufficient conditions which the scattering data must satisfy have been formulated by us here similarly to the way this is done (for the self-adjoint operator) in Appendix I to book (1). However, these conditions admit an equivalent formulation analogous to that given in the main text of that book.

If the function $s(\rho)$ is rational, then (as in the self-adjoint case) the basic equation (8) is an equation with a degenerate kernel and reduces to a system of linear algebraic equations. This circumstance makes it possible to construct very simply examples of operators having prescribed eigenvalues and spectral singularities of prescribed multiplicity.

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Note: Figure translations are in progress. See original paper for figures.