

# ON REPRESENTATIONS OF COMMUTATIVE SYMMETRIC BANACH ALGEBRAS AND COMMUTATIVE TOPOLOGICAL GROUPS IN THE SPACE $\Pi_k$

MATHEMATICS

1966

SovietRxiv

---

View the original and related papers at <https://sovietrxiv.org/items/ru-196601.18606>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

**Abstract**

**Full Text**

UDC 517 : 519.4 : 513.88

**MATHEMATICS**

**M. A. NAIMARK**

## ON REPRESENTATIONS OF COMMUTATIVE SYMMETRIC BANACH ALGEBRAS AND COMMUTATIVE TOPOLOGICAL GROUPS IN THE SPACE $\Pi_k$

*(Presented by Academician I. M. Vinogradov on 27 XII 1965)*

1. Let  $R$  be a commutative separable Banach algebra with identity  $e$ ; we shall call the algebra  $R$  **symmetric** if an involution  $x \rightarrow x^*$  is defined in  $R$  satisfying the conditions: 1)  $x^{**} = x$ ; 2)  $(\lambda x)^* = \bar{\lambda}x^*$ ,  $\lambda$  a number; 3)  $(x + y)^* = x^* + y^*$ ; 4)  $(xy)^* = x^*y^*$ ; 5)  $|x^*| = |x|$ . By a **representation** of the algebra  $R$  in the Pontryagin space  $\Pi_k$  we shall mean such a homomorphism  $x \rightarrow A_x$  of the algebra  $R$  into the algebra of bounded linear operators in  $\Pi_k$  that:  $\alpha$ )  $|A_x| \leq C|x|$  for all  $x \in R$  and some constant  $C$ , where  $|A_x|$  is the norm of the operator  $A_x$ ;  $\beta$ )  $A_{x^*} = (A_x)^*$ , where  $(A_x)^*$  is the operator adjoint to  $A_x$  in the sense of the indefinite scalar product  $(x, y)$  in  $\Pi_k$ .

The aim of the present article is to describe all representations in  $\Pi_k$  of the algebra  $R$ ; as a consequence one obtains a description of all unitary (in the indefinite sense) representations in  $\Pi_k$  of a commutative locally bicomact group  $G$  with a countable base of neighborhoods. We note that in the author's papers (<sup>1,2</sup>) a more general case of noncommutative algebras and groups was considered; however, the commutative case admits further detail.

In what follows the term "algebra" will denote a commutative symmetric separable Banach algebra with identity  $e$ ; moreover, everywhere below the space  $\Pi_k$  is assumed to be separable.

2. We first describe representations  $x \rightarrow A_x$  of the algebra  $R$  in an ordinary separable Hilbert space  $\mathfrak{H}$ , where now in  $\beta$ )  $(A_x)^*$  denotes the usual adjoint operator. Let  $T$  be the bicomact space of symmetric maximal ideals  $t$  of the algebra  $R$ , and let  $x(t)$  be the value of the element  $x \in R$  on the ideal  $t$ ; applying I, item 4 § 17 and Theorems 1 and 2 of item 3 (see the proof of Theorem 2) in (<sup>3</sup>), we arrive at the following result:

**Theorem 1.** *Every symmetric representation  $x \rightarrow A_x$  of the algebra  $R$  in a*

separable Hilbert space  $\mathfrak{H}$  is equivalent to a representation in the direct integral

$$\mathfrak{H} = \int_T \mathfrak{H}(t) d\sigma$$

with respect to some Borel measure  $\sigma$  on  $T$  of a  $\sigma$ -measurable family of separable Hilbert spaces  $\mathfrak{H}(t)$  and is given by the formula

$$A_x \{h(t)\} = \{x(t)h(t)\}$$

for  $\{h(t)\} \in \mathfrak{H}$ ,  $h(t) \in \mathfrak{H}(t)$ .

- Let now  $x \rightarrow A_x$  be a representation of the algebra  $R$  in the space  $\Pi_k$ . Put  $\mathfrak{A} = \{A_x : x \in R\}$ ; then  $\mathfrak{A}$  is a commutative symmetric algebra of operators in  $\Pi_k$  (see <sup>(4)</sup> or <sup>(5)</sup>). Its proper functionals (p.f.)  $\lambda_j(A_x)$  are called the p.f. of the representation  $x \rightarrow A_x$  and are denoted by  $\lambda_j(x)$ , while the hyperbolic subspace  $H$  of the algebra  $\mathfrak{A}$  is called the **hyperbolic subspace** of the representation  $x \rightarrow A_x$ . If  $H \neq (0)$ , then from the results of item 2 of <sup>(4)</sup> (see also item 3 of <sup>(5)</sup>) it follows:

**Theorem 2.** *The restriction of the representation of the algebra  $R$  to the hyperbolic subspace is the orthogonal sum of representations, each of whi-*

which are a direct sum of two mutually conjugate representations on two skew-related null subspaces.

Since  $\Pi_k = H \oplus H^\perp = H \dot{+} H^\perp$ , and the restriction of the representation  $x \rightarrow A_x$  has no non-real characters on  $H^\perp$ , it suffices to restrict ourselves to considering representations  $x \rightarrow A_x$  in  $\Pi_k$  with only real characters, which, evidently, may be regarded as points of the space  $T$ .

Let, in this case,  $\mathfrak{P}$  be some  $k$ -dimensional nonnegative subspace invariant with respect to all  $A_x$ ,  $x \in R$ . Put  $\mathfrak{M} = \mathfrak{P}^\perp$ ,  $\mathfrak{R} = \mathfrak{P} \cap \mathfrak{M}$ ; then  $\mathfrak{M}, \mathfrak{R}$  are also invariant with respect to all  $A_x$ ,  $x \in R$ ,  $\mathfrak{M}$  is a nonpositive, and  $\mathfrak{R}$  a null subspace in  $\Pi_k$ ; choose  $\mathfrak{P}$  so that  $\mathfrak{R}$  has maximal dimension. Let  $\mathfrak{R}'$  be a null subspace in  $\Pi_k$  skew-related to  $\mathfrak{R}$ ; put  $\mathfrak{H} = \mathfrak{M} \cap \mathfrak{R}'^\perp$ ,  $\Pi = \mathfrak{P} \cap \mathfrak{R}'^\perp$ . Then

$$\mathfrak{M} = \mathfrak{R} \oplus \mathfrak{H}, \quad \mathfrak{P} = \mathfrak{R} \oplus \Pi, \quad \Pi_k = (\mathfrak{R} \dot{+} \mathfrak{R}') \oplus \mathfrak{H} \oplus \Pi, \quad (1)$$

where  $\mathfrak{R}, \mathfrak{R}', \mathfrak{H}, \Pi$  are linearly independent,  $\Pi$  is positive,  $\mathfrak{H}$  is negative, i.e. a Hilbert space with scalar product  $[x, y] = -(x, y)$ . Let  $\mathfrak{S}_\lambda$  be the root lineal of the algebra  $\mathfrak{A}$  corresponding to the character  $\lambda$ , and let  $\lambda_1, \dots, \lambda_q$  be those among all the characters  $\lambda_1, \dots, \lambda_p$  for which  $\mathfrak{R}_j = \mathfrak{R} \cap \mathfrak{S}_{\lambda_j} \neq (0)$ ; then  $\mathfrak{R} = \sum_{j=1}^q \dot{+} \mathfrak{R}_j$ , and

- The functionals  $\lambda_1, \dots, \lambda_q$  do not depend on the choice of the subspace  $\mathfrak{P}$ , for which  $\mathfrak{R}$  has maximal dimension.

These characters  $\lambda_1, \dots, \lambda_q$  are called the **special** characters of the representation  $x \rightarrow A_x$ .

In each  $\mathfrak{R}_j$  one can choose a basis  $\xi_{j1}, \dots, \xi_{jr_j}$ ,  $r_j = \dim \mathfrak{R}_j$ , so that

$$A_x \xi_{jl} = \sum_{s=1}^l \lambda_{jls}(x) \xi_{js}, \quad \lambda_{jll}(x) = \lambda_j(x). \quad (2)$$

Formulas (2) define, for each  $j = 1, \dots, q$ , a representation of the algebra by triangular matrices with  $\lambda_j(x)$  on the diagonal.

Since  $\Pi$  is a finite-dimensional Hilbert space, it follows that

$$\Pi = \sum_{\nu=1}^p \oplus \Pi^\nu, \quad (3)$$

where  $\Pi^\nu = \{\xi : \xi \in \Pi, A_x \xi = \lambda_\nu(x) 1 + n \text{ for some } n \in \mathfrak{R}\}$  (some of the  $\Pi^\nu$  may be equal to (0)).

Let  $\{\eta_{jl}, l = 1, \dots, r_j, j = 1, \dots, q\}$  be a basis in  $\mathfrak{R}'$ , biorthogonal to  $\{\xi_{jl}\}$ . Using (1), Theorem 1, and repeating the arguments in (5) (see also (4)), we conclude that the following holds.

**Theorem 3.** *Every representation  $x \rightarrow A_x$  in a separable space  $\Pi_k$  of an algebra  $R$  with only real characters is given, in accordance with the decompositions (1), (3) and the bases  $\{\xi_{jl}\}, \{\eta_{jl}\}$ , by the formulas (2) and*

$$\mathfrak{H} = \int_T \mathfrak{H}(t) d\sigma,$$

$$A_x \{h(t)\} = - \sum_{j=1}^q \sum_{l=1}^{r_j} \left[ \int_T (h(t), h_{jl}(x, t)) d\sigma \right] \xi_{jl} + \{x(t)h(t)\}$$

for  $\{h(t)\} \in \mathfrak{H}$ ,

$$A_x \pi^\nu = \sum_{j=1}^q \sum_{l=1}^{r_j} (\pi^\nu, \pi_{jl}^\nu) \xi_{jl} + \lambda_\nu(x) \pi^\nu \quad \text{for } \pi^\nu \in \Pi^\nu, \nu = 1, \dots, p,$$

$$A_x \eta_{jl} = \sum_{\mu=1}^q \sum_{\nu=1}^{r_\mu} a_{j\mu\nu}(x) \xi_{\mu\nu} + \sum_{\nu=l}^{r_j} \overline{\lambda_{j\nu l}(x^*)} \eta_{j\nu} + \{h_{jl}(x^*, t)\} + \sum_{\nu=1}^p \pi_{jl}^\nu(x^*),$$

where  $\lambda_1, \dots, \lambda_q$  are special functionals of the representation;  $T$  is the space of symmetric maximal ideals of the algebra  $R$ ;  $\sigma$  is a Borel measure on  $T$ ;  $\mathfrak{H}(t)$  is a

$\sigma$ -measurable family of separable Hilbert spaces;  $\lambda_{jl_s}(x), \alpha_{jl_{\mu\nu}}(x)$  are numerical, and  $\{h_{jl}(x, t)\}, \pi_{jl}^\nu(x)$  are vector-functions on  $\overline{R}$  with values in  $\mathfrak{H}, \Pi^\nu$ , continuous on  $R$ , where  $\alpha_{jl_{\mu\nu}}(x^*) = \alpha_{\mu\nu jl}(\overline{x}), \lambda_j(x^*) = \overline{\lambda_j(x)}$ ,

$$h_{jl}(x, t) = (\overline{x(t)} - \overline{\lambda_j(x)})\xi_{jl}(t) - \sum_{\mu=l+1}^{r_j} \overline{\lambda_{j\mu l}(x)} \xi_{j\mu}(t), \quad j, l = 1, \dots, q,$$

$\sigma$ -almost everywhere for  $t \neq \lambda_j, j = 1, \dots, q$ ;

$$\pi_{jl}^\nu(x) = [\overline{\lambda_\nu(x)} - \overline{\lambda_j(x)}]\pi_{jl}^\nu - \sum_{\mu=l+1}^{r_j} \overline{\lambda_{j\mu l}(x)} \pi_{j\mu}^\nu$$

for  $j \neq \nu$ ;  $\{\xi_{j\mu}(t)\}$  is a  $\sigma$ -measurable vector-function on  $T$  with values  $\xi_{j\mu}(t) \in \mathfrak{H}(t)$ ;  $\pi_{jl}^\nu$  are vectors from  $\Pi^\nu$ .

4. Let  $G$  be a locally bicomact commutative group with a countable base of neighborhoods, and let  $g \rightarrow U_g$  be a unitary representation of the group  $G$  in  $\Pi_k$ . Put  $\omega(g) = |U_g|$ . Let  $L_\omega^1(G)$  be the totality of all such measurable functions  $x(g)$  on  $G$  that  $|x| = \int |x(g)|\omega(g) dg < \infty$ . From the relations  $\omega(g_1 g_2) \leq \omega(g_1)\omega(g_2), \omega(g^{-1}) = |U_{g^{-1}}| = |U_g^*| = |U_g| = \omega(g)$ , it easily follows that  $L_\omega^1(G)$  is a symmetric algebra, with the usual definition of addition and multiplication by a scalar, with convolution multiplication and involution  $x^*(g) = \overline{x(g^{-1})}$ . Since  $G$  has a countable base of neighborhoods,  $L_\omega^1(G)$  is separable.

Let  $R_\omega$  be the algebra obtained from  $L_\omega^1(G)$  by adjoining an identity. Then the results of §§ 2, 3 are applicable to  $R_\omega$ . Applying the usual arguments, we conclude that the maximal ideals  $t$  in  $R_\omega$  distinct from  $t_0 = L_\omega^1(G)$  are given by the relations

$$x^\wedge(t) = \int x(g)\chi(g) dg, \quad x^\wedge(t_0) = 0, \quad (4)$$

where  $\chi(g)$  is a continuous function on  $G$  satisfying the conditions  $\chi(e) = 1$  ( $e$  is the identity of the group  $G$ ),  $\chi(g_1 g_2) = \chi(g_1)\chi(g_2), |\chi(g)| \leq C\omega(g)$ , where  $C$  is some constant; moreover, the correspondence  $t \rightarrow \chi$  is thus established one-to-one. The ideal  $t \rightarrow \chi$  is symmetric if and only if  $\chi(g^{-1}) = \overline{\chi(g)}$ , i.e.  $|\chi(g)| = 1$ , i.e.  $\chi$  is a character of the group  $G$ . Hence it easily follows that the space  $T$  of symmetric maximal ideals of the algebra  $R_\omega$  is homeomorphic to the space  $G_\infty^*$ , the group  $G^*$  of characters of the group  $G$  with an infinite point adjoined. We shall therefore identify  $T$  with  $G_\infty^*$ . To the representation  $g \rightarrow U_g$  of the group  $G$  there corresponds the representation  $y \rightarrow A_y$  of the algebra  $R_\omega$  by the formula  $A_y = \lambda 1 + \int x(g)U_g dg$  for  $y = \lambda e + x, x \in L_\omega^1(G)$ . Special f., special s.f. and the

hyperbolic subspace  $H$  of the representation  $y \rightarrow A_y$  will respectively be called the s.f., special s.f. and hyperbolic subspace of the representation  $g \rightarrow U_g$ .

II. If  $H \neq (0)$ , the assertion of Theorem 2 is valid for a unitary representation of the group  $G$  in  $\Pi_k$ .

Considering the representation  $g \rightarrow U_g$  in  $H^\perp$  instead of  $\Pi_k$ , we may henceforth assume that this representation has only real s.f.  $\lambda_1, \dots, \lambda_p$ ; in view of the identification established above of  $T$  with  $G_\infty^*$ , they are characters  $\lambda_j = \chi_j \in G^*$ ,  $j = 1, \dots, p$ . In particular, the special s.f.  $\lambda_j = \chi_j$ ,  $j = 1, \dots, q$ , of the representation are called its special characters. It is not hard to verify that

$$A_x^{(1)} = \int x(g)U_g^{(1)} dg, \quad \lambda_{jls}(x) = \int x(g)\lambda_{jls}(g) dg,$$

etc. for  $x(g) \in L_\omega^1(G)$ , where  $g \rightarrow U_g^{j(1)}$  is an ordinary continuous unitary representation of the group  $G$  in  $\mathfrak{H}$ , and  $\lambda_{jls}(g)$  is a continuous numerical function on  $G$ . Taking into account (4) and these formulas and applying Theorem 3, we arrive at the following result:

**Theorem 4.** Let  $G$  be a commutative locally bicomact group with a countable base of neighborhoods. Every continuous unitary representation  $g \rightarrow U_g$  of the group  $G$  in a separable space  $\Pi_k$  with only real c.f. is given, in accordance with the decompositions (1) and (3) and the bases  $\{\xi_{jl}\}, \{\eta_{jl}\}$ , by the formulas

$$\mathfrak{H} = \int_{G^*} \mathfrak{H}(\chi) d\sigma,$$

$$U_g \xi_{jl} = \sum_{s=1}^l \lambda_{jls}(g) \xi_{js}, \quad \lambda_{jll}(g) = \chi_j(g), \quad j = 1, \dots, q; \quad l = 1, \dots, r_q,$$

$$U_g \{h(\chi)\} = - \sum_{j=1}^q \sum_{l=1}^{r_j} \left[ \int_G (h(\chi), h_{jl}(g, \chi)) d\sigma \right] \xi_{jl} + \{\chi(g)h(\chi)\},$$

$$U_g \pi^\nu = \sum_{j=1}^q \sum_{l=1}^{r_j} (\pi^\nu, \pi_{jl}^\nu(g)) \xi_{jl} + \chi_\nu(g) \pi^\nu \quad \text{for } \pi^\nu \in \Pi^\nu, \quad \nu = 1, \dots, p,$$

$$U_g \eta_{jl} = \sum_{\mu=1}^q \sum_{\nu=1}^{r_\mu} \alpha_{j\mu\nu}(g) \xi_{\mu\nu} + \sum_{\nu=l}^{r_j} \overline{\lambda_{j\nu l}(g^{-1})} \eta_{j\nu} + \{h_{jl}(g^{-1}, t)\} + \sum_{\nu=1}^p \pi_{jl}^\nu(g^{-1}),$$

where  $\chi_1, \dots, \chi_q$  are distinguished characters of the representation;  $\sigma$  is a Borel measure on  $G^*$ ;  $\mathfrak{H}(\chi)$  is a measurable family of separable Hilbert spaces;

$\lambda_{jl_s}(g), \alpha_{jl_{\mu\nu}}(g)$  are numerical functions, and  $h_{jl}(g, t), \pi_{jl}^\nu(g)$  are vector-functions on  $G$  with values in  $\mathfrak{H}$  and  $\Pi^\nu$ , continuous on  $G$ , with

$$\alpha_{jl_{\mu\nu}}(g^{-1}) = \overline{\alpha_{\mu\nu jl}(g)};$$

$$h_{jl}(g, t) = (\overline{\chi(g)} - \overline{\chi_j(g)})\xi_{jl}(t) - \sum_{\mu=l+1}^{r_j} \overline{\lambda_{j\mu l}(g)}\xi_{j\mu}(t), \quad j, l = 1, \dots, q,$$

$\sigma$ -almost everywhere for  $\chi \neq \chi_j, j = 1, \dots, q$ ;

$$\pi_{jl}^\nu(g) = [\overline{\chi_\nu(g)} - \overline{\chi_j(g)}]\pi_{jl}^\nu - \sum_{\mu=l+1}^{r_j} \overline{\lambda_{j\mu l}(g)}\pi_{j\mu}^\nu, \quad j, l = 1, \dots, q,$$

for  $j \neq \nu$ ;  $\{\xi_{j\mu}(\chi)\}$  is a  $\sigma$ -measurable vector-function on  $G^*$  with values  $\xi_{j\mu}(\chi) \in \mathfrak{H}(\chi)$ ;  $\pi_{jl}^\nu$  are vectors from  $\Pi_\nu$ .

For the cases in which  $G$  is the additive group of the integers or the additive group of all real numbers, Theorem 4 apparently yields the known results of I. S. Iokhvidov and M. G. Krein (see, for example, <sup>(6)</sup>); for the case in which  $G$  is the additive  $n$ -dimensional group, close results were obtained in an unpublished work of A. I. Shtern.

Mathematical Institute named after V. A. Steklov  
Academy of Sciences of the USSR

Received  
15 XII 1965

## CITED LITERATURE

- <sup>1</sup> M. A. Naimark, *Izv. AN SSSR, ser. matem.*, **30**, No. 5 (1966).
- <sup>2</sup> M. A. Naimark, *Acta Sci. Math. Szeged.*, **26**, No. 3-4, 201 (1965).
- <sup>3</sup> M. A. Naimark, *Normed rings*, Moscow, 1958.
- <sup>4</sup> M. A. Naimark, *DAN*, **161**, No. 4, 767 (1965).
- <sup>5</sup> M. A. Naimark, *Math. Ann.*, **162**, 1, 147 (1965).
- <sup>6</sup> I. S. Iokhvidov, M. G. Krein, *Tr. Moscow Math. Soc.*, **8**, 413 (1959).

*Note: Figure translations are in progress. See original paper for figures.*

*Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.*