ON THE CAUCHY PROBLEM FOR DEGENERATE HYPERBOLIC EQUATIONS OF SECOND ORDER

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Abstract

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MATHEMATICS

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ON THE CAUCHY PROBLEM FOR DEGENERATE HYPERBOLIC EQUATIONS OF SECOND ORDER

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In works devoted to the study of the correctness of the Cauchy problem for a hyperbolic equation of second order with initial data on a line of parabolic degeneration, much attention has been paid to the following problem (see ([1],[2])):

$$-\Delta^2(x,y)u_{xx} + u_{yy} = au_x + bu_y + cu + f \quad (y>0,\ 0\le x\le 1); \tag{1}$$

$$u(x,+0) = \mu(x), \qquad u_y(x,+0) = v(x) \quad (0\le x\le 1), \tag{2}$$

where

$$\Delta(x,y) = \omega\Delta(y), \qquad 1\le\omega\le \mathrm{const}, \qquad \Delta(y)>0\ (y>0),$$

$$\Delta(+0)=0, \qquad \Delta'\ge 0. \tag{3}$$

L. Bers showed ([3]) that problem (1)–(2) is correct in the classical sense if the lower-order terms are absent from equation (1). Later it was noted that restrictions need be imposed only on the coefficient $a(x,y)$. The following (apparently, up to now the most general) condition for correctness was established by Protter ([4])*:

$$a_0(x) = \overline{\lim}_{y\to+0}\frac{y\,|a(x,y)|}{\Delta(y)} = 0 \quad (0\le x\le 1). \tag{4}$$

However, even in the case of a power-law decrease of the function $\Delta(y)$, this condition turns out to be stringent.

In particular, for $\Delta(x,y)=y$ Hellwig showed ([5]) that the estimate $a_0<2$ is sufficient, and for $\Delta(x,y)=y^\alpha$ $(\alpha>0)$ Chi Min-yu ([6]) indicates the estimate

$a_0 < \alpha$. From recent results of S. A. Tersenov ([7]) it follows that if $\Delta(x,y) = y^2$ $(\alpha > 0)$, $a_0 \leq \text{const}$, and all the parameters of problem (1)–(2) are differentiable with respect to x a sufficient number of times (depending on a_0), then this problem is posed correctly. As Tellerstedt ([8]) and I. S. Berezin ([9]) showed, problem (1)–(2) may turn out to be incorrectly posed (stability of the solution in the uniform metric is violated) if the coefficient a is not subjected to restrictions. Works ([11–13]) are also devoted to this circle of questions.

In the present note the above-mentioned criteria of correctness are refined and generalized.

1°. Consider the Cauchy problem

$$Au_{xx} + 2Bu_{xy} + u_{yy} = au_x + bu_y + cu + f \quad (y > 0,\ 0 \leq x \leq 1); \tag{5}$$

$$u(x, +0) = \mu(x), \qquad u_y(x, +0) = v(x) \quad (0 \leq x \leq 1), \tag{6}$$

where

$$\Delta^2(x,y) = B^2 - A > 0 \ (y > 0), \qquad \Delta(x, +0) \geq 0, \tag{7}$$

i.e., on the line $y = 0$ degeneration of type is possible.

* As analysis shows, in paper ([4]), contrary to the author' s assertion, criterion (4) is proved only in the case $\Delta(y) \geq \text{const} \cdot y^\alpha$ $(\alpha > 0)$.

By D we shall denote the open characteristic triangle based on the segment $(y = 0,\ 0 \leq x \leq 1)$. For an arbitrary function $\varphi(x,y)$, continuous for $(x,y) \in D$, we adopt the notation

$$\varphi^*(y) = \max_x |\varphi(x,y)|. \tag{8}$$

We shall also denote

$$\alpha_1 = a - bB + B_y + \Delta_y + B(B_x + \Delta_x), \qquad \alpha_2 = \alpha_1 - 2(\Delta_y + B\Delta_x),$$

$$2\Delta a = |\alpha_1| + |\alpha_2|. \tag{9}$$

Theorem 1. Let the functions A, B, and f be continuously differentiable in $\overline{D}$ with respect to x $2p+3$ times $(p \geq 0)$, and let the functions ν and μ be, respectively, $2p+4$ and $2p+5$ times; in addition, let A and B be differentiable in D with respect to y, and let the functions

$$\left(\frac{\partial^k}{\partial x^k}a\right)^*, \qquad \left(\frac{\partial^k}{\partial x^k}b\right)^* \quad \text{and} \quad y\left(\frac{\partial^k}{\partial x^k}C\right)^* \quad (k = 0, 1, \dots, 2p+2)$$

be integrable for $y \geq 0$.

If for some i $(= 1, 2)$

$$\int_0^y t\Delta^*(t)\, d\left\{f_p^i(t)\exp\left(\int_t^y \alpha^*(\tau)\, d\tau\right)\right\} < +\infty \quad (y > 0), \tag{10}$$

where

$$f_p^i(y) = \int_0^y \alpha_i^*(t_1)(y - t_1)\int_0^{t_1} \alpha_i^*(t_2)(t_1 - t_2)\cdots\int_0^{t_p} \alpha_i^*(t_{p+1})\, dt_{p+1}\cdots dt_1, \tag{11}$$

then problem (5)–(6) has a unique solution u, possessing continuous derivatives in $\overline{D}$

$$\frac{\partial^k}{\partial x^{k-i}\partial y^i}u \quad (i = 0, 1, 2;\ i \leq k \leq p+2).$$

This solution is stable in the following sense: let the functions f_i, μ_i, ν_i $(i = 1, 2)$ correspond to the solutions u_i. Then for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that from the estimate

$$\sum_{k=0}^{2p+3}\max_{\overline{D}}\left|\frac{\partial^k}{\partial x^k}(f_1 - f_2)\right| + \sum_{k=0}^{2p+4}\max_x\left|\frac{\partial^k}{\partial x^k}(\nu_1 - \nu_2)\right| + \sum_{k=0}^{2p+5}\max_x\left|\frac{\partial^k}{\partial x^k}(\mu_1 - \mu_2)\right| < \delta \tag{12}$$

it follows that

$$\sum_{i=0}^{2}\sum_{k=i}^{p+2}\max_D\left|\frac{\partial^k(u_1 - u_2)}{\partial x^{k-i}\partial y^i}\right| < \varepsilon. \tag{13}$$

2°. Fix some domain Ω of the complex plane $z = x + it$, containing the segment $(t = 0,\ 0 \leq x \leq 1)$. A continuous function $\varphi(x, y)$ in $\overline{D}$ will be assigned to the class $H(D, \Omega)$ if, for each fixed y, it admits an analytic continuation in x to the domain Ω.

Theorem 2. If the functions $A, B, \Delta, a, b, c, f, \mu, \nu$ belong to the class $H(D, \Omega)$, then problem (5)–(6) has a unique solution $u(x, y) \in H(D, \Omega_1)$, where the domain Ω_1 contains the segment $(t = 0,\ 0 \leq x \leq 1)$ and depends only on Ω.

This solution is stable in the following sense: let the functions f_i, μ_i, ν_i $(i = 1, 2)$ correspond to the solutions u_i. Then for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that if

$$|f_1(z,y) - f_2(z,y)| + |\mu_1(z) - \mu_2(z)| + |\nu_1(z) - \nu_2(z)| < \delta, \qquad z \in \Omega,\ y \geq 0, \tag{14}$$

then

$$|u_1(z,y) - u_2(z,y)| < \varepsilon, \qquad z \in \Omega,\ y \geqslant 0. \tag{15}$$

The example of I. S. Berezin mentioned above ([9]) shows that, from estimate (14), when $\operatorname{Im} z = 0$, in general estimate (15) for $\operatorname{Im} z = 0$ does not follow. At the same time, from this example (analogous to Hadamard’ s well-known example for the Laplace equation) and from Theorem 2 it follows that problem (5)–(6) may possess properties inherent in the Cauchy problem for an elliptic equation.

3°. Theorem 1 strengthens all the above-mentioned criteria for the correctness of problem (1)–(2). Indeed, let

$$\overline{\lim_{y \to +0}} \frac{|a(x,y)|}{\Delta'(y)} \leq q < +\infty \qquad (0 \leq x \leq 1). \tag{16}$$

Then estimate (10) is certainly satisfied if

$$\int_0^y t^{p+2} [\Delta(t)]^{1+p-q} \Delta'(t)\, dt < +\infty \qquad (p \geqslant 0). \tag{17}$$

Thus, problem (1)–(2) is correct in the sense of Theorem 1 if $q < p + 2$. For $\Delta(y) = y^\alpha$ $(\alpha > 0)$, it suffices to have the estimate $a_0 < (\alpha + 1) \times (p + 2)$ (in the notation of (4)). Protter’ s criterion ([4]), also, evidently, strengthened by Theorem 1, turns out to be the more stringent the faster $\Delta(y)$ tends to zero. For example, for $\Delta(y) = \exp\{-y^{-\beta}\}$ $(\beta > 0)$, according to criterion (17), it suffices to have the estimate

$$\overline{\lim_{y \to +0}} \frac{y^{\beta+1} |a(x,y)|}{\Delta(y)} < \beta(p + 2) \qquad (0 \leq x \leq 1). \tag{18}$$

We also note that if the function $\{\min_x \Delta\}^{-1}$ is integrable for $y \geqslant 0$, then the coefficients of equation (5) are free from restrictions, and even its right-hand side may be taken to be a nonlinear function satisfying the usual Lipschitz conditions in such cases.

4°. In the work of Chi Min-yu ([10]) the general solution of the equation

$$-y^2 u_{xx} + u_{yy} = a u_x \qquad (y > 0) \tag{19}$$

for $a = \text{const}$ was obtained in explicit form. In the special case when $a = 4n+1$ ($n \geqslant 0$ an integer), the Cauchy problem with initial data (2) for $\nu \equiv 0$ has the unique solution

$$u = \sum_{k=0}^{n} c_k y^{2k} \mu^{(k)}\left(x + \frac{y^2}{2}\right)$$

(c_k are constants). Hence it follows that, for large a, problem (19)–(2) with $\nu \equiv 0$ has only a generalized solution if the function μ is not infinitely differentiable. At the same time, this formula gives an explicit dependence of the character of stability of the solution on the magnitude of a.

A direct application to this problem of the criterion indicated in the preceding paragraph gives somewhat more restrictive conditions of correctness; however, this same example shows that Theorem 1 cannot be substantially improved.

5°. The results obtained carry over to the following Cauchy problem for the system

$$u_{iy} + A_i u_{1x} + B_i u_{2x} = a_i u_1 + b_i u_2 + f_i,$$

$$u_i(x, +0) = \mu_i(x) \qquad (i = 1, 2;\ y > 0;\ 0 \leq x \leq 1), \tag{20}$$

where

$$\Delta^2(x, y) = (A_1 - B_2)^2 + 4A_2 B_1 > 0 \quad (y > 0), \qquad \Delta(x, +0) \geqslant 0. \tag{21}$$

A special case of the problem was studied by S. A. Tersenov ([7]).

Obviously, one may assume that $A_1 \geqslant B_2$. In the case $A_2 \equiv B_1 \equiv 0$ correctness is obvious. Let us additionally suppose that for $y > 0$, $A_2 > 0$. Denote

$$c = A_1 - B_2 + \Delta, \quad 2d = A_1 + B_2, \quad 2A_2 e = c,$$

$$\alpha_1 = b_1 + e_y + d e_x + e(a_1 - e a_2 - b_2), \tag{22}$$

$$\omega A_2 = \Delta, \quad \alpha_2 = \alpha_1 + \omega_y + d\omega_x, \quad \omega\alpha = |\alpha_1| + |\alpha_2|.$$

Theorem 1 remains valid, with the corresponding changes, if the functions A_i, B_i, f_i and c are continuous in $\overline{D}$ together with their derivatives with respect to x up to order $(2p+3)$ $(p \geqslant 0)$, μ_i up to order $(2p+4)$, A_i, B_i are continuously differentiable in D with respect to y, the functions $\left(\frac{\partial^k}{\partial x^k} a_i\right)^*$ and $\left(\frac{\partial^k}{\partial x^k} b_i\right)^*$ $(i = 1, 2;\ k = 0, 1, \dots, 2p+2)$ (see notation (8)) are integrable for $y \geqslant 0$, and for some i $(= 1, 2)$

$$\int_0^y t\omega^*(t)\, d\left\{ f_p^i(t) \exp\left(\int_t^y \alpha^*(\tau)\, d\tau \right) \right\} < +\infty \quad (y > 0), \tag{23}$$

where

$$f_p^i(y) = \int_0^y A_2^*(t_1) \int_0^{t_1} \alpha_i^*(t_2) \cdots \int_0^{t_{2p}} A_2^*(t_{2p+1}) \int_0^{t_{2p+1}} \alpha_i^*(t_{2p+2})\, dt_{2p+2} \cdots dt_1. \tag{24}$$

As with problem (5)–(6), problem (20) is correct with nonlinear right-hand sides if the function $\{\min_x \omega\}^{-1}$ is integrable for $y \geqslant 0$.

Finally, we note that Theorem 2 remains valid also for problem (20), if the functions A_i, B_i, a_i, b_i, f_i $(i = 1, 2)$, c, d, e, and ω belong to the class $H(D, \Omega)$.

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Note: Figure translations are in progress. See original paper for figures.