ON AN INTEGRAL REPRESENTATION OF FUNCTIONS OF EXPONENTIAL GROWTH

1966

SovietRxiv

Abstract

Full Text

MATHEMATICS

M. A. EVGRAFOV

ON AN INTEGRAL REPRESENTATION OF FUNCTIONS OF EXPONENTIAL GROWTH

(Presented by Academician M. V. Keldysh, 31 VIII 1965)

An entire function of exponential growth in two complex variables is a function $v(z_1, z_2)$ satisfying, for all z_1 and z_2, the inequality

$$|v(z_1, z_2)| < M \exp\{a(|z_1| + |z_2|)\}.$$

We shall deal not only with entire functions of exponential growth. The class of functions that interests us is easily singled out if in the space (z_1, z_2) one introduces "polar coordinates" z and φ by the formulas $z_1 = z\cos\varphi,\ z_2 = z\sin\varphi$. Namely, we shall say that a function $v(z,\varphi)$ is a function of exponential type, equal to σ, in the half-space $\operatorname{Re} z > 0$, if it is regular for $\operatorname{Re} z > 0$ and for all complex φ, is periodic in φ with period 2π, and satisfies the inequality

$$|v(z,\varphi)| < M_\varepsilon \exp\{(\sigma+\varepsilon)|z| \exp|\operatorname{Im}\varphi|\} \qquad (\operatorname{Re} z > 0) \tag{1}$$

for any $\varepsilon > 0$.

It is clear that one may also speak of entire functions of exponential type σ. It is easy to verify that if a function $w(z_1, z_2)$ is an entire function of exponential growth in the usual sense, then the function $v(z,\varphi) = w(z\cos\varphi, z\sin\varphi)$ is an entire function of exponential type in the sense of our definition. Conversely, if the function $v(z,\varphi) = w(z\cos\varphi, z\sin\varphi)$ is an entire function of exponential type in the sense of our definition, then the function $w(z_1, z_2)$ need not be an entire function of exponential growth in the usual sense. True, it can be represented in the form

$$w(z_1, z_2) = w_1(z_1, z_2) + \sqrt{z_1^2 + z_2^2}\, w_2(z_1, z_2),$$

where w_1 and w_2 are entire functions of exponential growth.

In the present note the following integral representation for functions of exponential type will be proved:

Theorem 1. Let $v(z,\varphi)$ be a function of exponential type, of type σ, in the half-space $\operatorname{Re} z > 0$. Denote by V_a the semi-infinite cylinder

$$x > -a, \qquad x_1^2 + x_2^2 < a^2,$$

in the space $i(x, x_1, x_2)$, and

$$E = E(z, \varphi; x, x_1, x_2) = \frac{1}{8\pi^3}\,\frac{\exp\{-z[x + i(x_1\cos\varphi + x_2\sin\varphi)]\}}{z}.$$

If S_ε denotes the surface of the cylinder $V_{\sigma+\varepsilon}$, then for the function $v(z, \varphi)$ the integral representation

$$v(z, \varphi) = \iint_{S_\varepsilon} \left(\mu\,\frac{\partial E}{\partial n} - E\,\frac{\partial \mu}{\partial n}\right) ds \tag{2}$$

holds.

for any $\varepsilon > 0$. Here $\partial/\partial n$ denotes differentiation along the normal to the surface of integration, and $\mu = \mu(x, x_1, x_2)$ is a function harmonic outside the cylinder V_σ and tending to zero when the point (x, x_1, x_2) tends to infinity outside the cylinder $V_{\sigma+\varepsilon}$.

For entire functions a stronger assertion is valid.

Theorem 2. Let $v(z, \varphi)$ be an entire function of exponential type σ. Denote by V_a' the finite cylinder

$$-a < x < a, \qquad x_1^2 + x_2^2 < a^2.$$

If S_ε denotes the surface of the cylinder $V_{\sigma+\varepsilon}'$, then for the function $v(z, \varphi)$ the integral representation (2) is valid with a function μ harmonic outside the finite cylinder V_σ and tending to zero as $(x, x_1, x_2) \to \infty$.

If one uses a finer characteristic of the growth of the function $v(z, \varphi)$ than its type σ, one can refine the dimensions of the domain of harmonicity of the function μ.

We introduce the indicator $H_v(\varphi)$ of the function $v(z, \varphi)$, characterizing its growth, as follows:

$$H_v(\varphi) = \varlimsup_{s\to\infty} \frac{\ln\{\,|v(iy, \varphi + i\alpha)| + |v(-iy, \pi + \varphi + i\alpha)|\,\}}{|y|\operatorname{ch}\alpha} \qquad (0 \leqslant \varphi < 2\pi)$$

(here $s = y^2 + \alpha^2$). By $D_\varepsilon(v)$ we denote the domain in the plane (x_1, x_2) that is the common part of all half-planes (here $\zeta = x_1 + ix_2$)

$$\operatorname{Re}(\zeta e^{-i\varphi}) < H_v(\varphi) + \varepsilon, \qquad 0 \leqslant \varphi < 2\pi.$$

Theorem 3. Let $v(z,\varphi)$ be a function of exponential type equal to σ, with indicator $H_v(\varphi)$. Denote by $\widetilde{V}_\varepsilon$ the semi-infinite cylinder

$$x > -(\sigma+\varepsilon), \qquad (x_1,x_2) \in D_\varepsilon(v).$$

Then for the function $v(z,\varphi)$ the integral representation (2) is valid with the cylinder $V_{\sigma+\varepsilon}$ replaced by the cylinder $\widetilde{V}_\varepsilon$.

We shall briefly set forth the main points of the proof of Theorems 1–3. Put

$$\mu(x,x_1,x_2) = \int_0^\infty \int_0^{2\pi} v(z,\varphi) e^{xz+iz(x_1\cos\varphi+x_2\sin\varphi)}\, z\, dz\, d\varphi. \tag{3}$$

From the fact that $v(z,\varphi)$ is a function of exponential type σ in the half-plane $\operatorname{Re} z > 0$, it follows that the integral on the right-hand side of formula (3) converges uniformly (and is bounded) for $x \leqslant -(\sigma+\varepsilon)$ and for arbitrary x_1 and x_2. It is also clear that as $x \to -\infty$ this integral tends to zero. By differentiation it is easily established that $\mu(x,x_1,x_2)$ is a harmonic function in the half-space $x < -\sigma$.

We shall show that the function $\mu(x,x_1,x_2)$ can be analytically continued from this half-space to the entire space except for the cylinder V_σ. For this purpose we introduce in the space (x,x_1,x_2) cylindrical coordinates x,ρ,θ, i.e., set $x_1 = \rho\cos\theta$, $x_2 = \rho\sin\theta$, and write the integral for μ in the form

$$\mu(x,\rho\cos\theta,\rho\sin\theta) = \int_0^\infty \int_0^{2\pi} v(z,\theta+\varphi) e^{z(x+i\rho\cos\varphi)}\, z\, dz\, d\varphi. \tag{4}$$

Next denote by $L_{\delta,R}$, where $R \geqslant 0$, $0<\delta<\pi/2$, the broken line with vertices at the points (written in the order of traversal) $0,\ \delta,\ \delta+iR,\ \pi-\delta+iR,\ \pi-\delta,\ \pi+\delta,\ \pi+\delta-iR,\ 2\pi-\delta-iR,\ 2\pi-\delta,\ 2\pi$. Taking into account the regularity of $v(z,\varphi)$ with respect to φ and its periodicity, the inner in-

the integral in formula (4) may be taken not over the interval $(0,2\pi)$, but over any broken line $L_{\delta,R}$. Further, in view of the fact that $v(z,\varphi)$ is a function of exponential type, for sufficiently large $-x$ and ρ we may pass to the limit as $R \to +\infty$. This will lead us to the formula

$$\mu = I_1 + I_2,$$

$$I_k = \int_0^\infty \int_{L_k} v(z,\theta+\varphi) e^{z(x+i\rho\cos\varphi)} z\, dz\, d\varphi \qquad (k=1,2),$$

where L_1 is the broken line $(-\delta - i\infty, -\delta, \delta, \delta + i\infty)$, and L_2 is the broken line $(\pi - \delta + i\infty, \pi - \delta, \pi + \delta, \pi + \delta - i\infty)$.

In the integral I_1 we rotate the ray of integration through the angle $+\pi/2$, and in the integral I_2 through the angle $-\pi/2$. This is legitimate for sufficiently large $-x$ and ρ. After this one may put $\delta = 0$, and we obtain

$$\mu(x, \rho\cos\theta, \rho\sin\theta) = \int_0^\infty \int_{-\infty}^\infty q(x, y, \theta + i\alpha) e^{-\rho y \operatorname{ch} \alpha} y \, dy \, d\alpha, \tag{5}$$

where

$$q(x, y, \psi) = \frac{1}{i} \left[v(iy, \psi) e^{ixy} - v(-iy, \pi + \psi) e^{-ixy} \right].$$

The integral standing on the right-hand side of formula (5) converges uniformly and is bounded for $\rho \geq \sigma + \varepsilon$ and for any x. It is also clear that the integral tends to zero as $\rho \to +\infty$ (it is not difficult to show that it tends to zero also for fixed ρ, as $x \to \pm\infty$). For sufficiently large $-x$ and ρ, formulas (3) and (5) define one and the same harmonic function; hence formula (5) gives the analytic continuation of the function μ to the exterior of the cylinder $\rho > \sigma$, while both formulas give it in the exterior of the cylinder V_σ.

Let us proceed to obtaining the integral representation.

It is easy to show that the function $\mu(x, x_1, x_2)$, harmonic outside the cylinder $V_{\sigma+\varepsilon}$, can be expressed in terms of its values on the surface of this cylinder by means of Green' s formula

$$\mu = \frac{1}{4\pi} \iint_{S_\varepsilon} \left\{ \mu \frac{\partial}{\partial n} \left(\frac{1}{r} \right) - \frac{1}{r} \frac{\partial \mu}{\partial n} \right\} ds$$

(here $r = \sqrt{(x - t)^2 + (x_1 - t_1)^2 + (x_2 - t_2)^2}$).

On the other hand, from formula (3) it is clear that the function μ, as a function of the variables x_1 and x_2, is the Fourier transform of the function $v(z, \varphi) e^{xz}$ (with respect to the variables $z\cos\varphi$ and $z\sin\varphi$). Applying the inverse Fourier transform and changing the order of integration, we arrive at formula (2), since

$$\frac{1}{(2\pi)^2} \int_{-\infty}^\infty \int \frac{1}{4\pi r} e^{-iz(x_1\cos\varphi + x_2\sin\varphi)} \, dx_1 \, dx_2 = E(z, \varphi; t, t_1, t_2).$$

This argument proves Theorem 1.

To prove Theorem 2 one must also carry out (by the same method, but in the reverse order) the analytic continuation of the function μ into the half-space $x > \sigma$.

To prove Theorem 3, formula (5) should be transformed to the form

$$\mu(x, \rho\cos\theta, \rho\sin\theta) = \int_0^\infty \int_{-\infty}^\infty q(x, y, \theta_0 + i\alpha) e^{-\rho y \operatorname{ch}[\alpha + i(\theta - \theta_0)]} y\, dy\, d\alpha$$

(by shifting the line of integration in the inner integral, which is quite legitimate). The domain of convergence of the last integral is the half-space $\rho\cos(\theta - \theta_0) > H_v(\theta_0)$. This gives the assertion of Theorem 3.

In conclusion, let us make a few remarks.

First of all, we point out that the result is easily generalized to the case of n variables.

Let us also note that formulas (2) and (3) may be regarded as mutually inverse formulas of a certain integral transformation. This integral transformation establishes a one-to-one correspondence between certain classes of functions of n complex variables and complex-valued harmonic functions of $n+1$ real variables.

We note that analogous integral transformations can be constructed, establishing a one-to-one correspondence between functions of n complex variables and solutions of other elliptic equations (or systems) with constant coefficients. This generalization is more complicated, but it also presents no particular difficulties.

A considerably more difficult problem is the construction of analogous integral transformations for elliptic equations with variable coefficients.

Finally, let us point out an analogue of the integral transformation found here in the case where $n = 1$. If, instead of the Laplace equation, one takes the Cauchy-Riemann system, one obtains the one-sided Laplace transform.

Received
17 VIII 1965