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In work (1), using as an example the flow in a cylindrical tube of a Newtonian fluid with a strong (nonlinear) dependence of viscosity on temperature, it was shown that a phenomenon analogous to a thermal explosion is possible in the flow of a chemically inert fluid.

It is of interest to investigate the possibility of a hydrodynamic thermal "explosion" in the flow of a chemically inert non-Newtonian fluid obeying the power law (2),

$$\tau = k(dv/dy)^n, \quad (1)$$

where τ is the shear stress; dv/dy is the velocity gradient; k and n are quantities which, in the general case, depend on temperature. As is known, many polymer solutions and melts obey this law. In this case k depends strongly on temperature, while the exponent n depends weakly (3). For the most frequently encountered pseudoplastic fluids, $n < 1$.

The rheological equation is usually written in the form (1). But, so that the left-hand side always corresponds to the right-hand side, it should be written in the form

$$\tau = k|dv/dy|^n \operatorname{sign} dv/dy. \quad (2)$$

Let us consider the problem of steady axisymmetric flow of a non-Newtonian fluid, whose rheological state is described by relation (2), in an infinite tube of radius r_0. The flow takes place under the action of a constant pressure gradient; the density of the fluid is assumed constant. In view of the weak dependence of the exponent n on temperature, we shall regard it as constant. We shall take the dependence of k on temperature in the form

$$k = k_0 e^{U/RT}, \quad (3)$$

where k_0 and U are constants; R is the gas constant; T is the absolute temperature.

The system of equations of motion in stresses and of heat conduction, with allowance for energy dissipation, is written in the form

$$\frac{d\tau}{dr} + \frac{\tau}{r} - \frac{dP}{dz} = 0, \qquad \frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} + \frac{k}{\lambda J}\left|\frac{dv}{dr}\right|^{n+1} = 0. \tag{4}$$

Boundary conditions:

$$\text{for } r = 0 \quad dv/dr = 0, \quad dT/dr = 0; \qquad \text{for } r = r_0 \quad v = 0, \quad T = T_0. \tag{5}$$

Here v is the velocity; $dP/dx = b$ is the pressure gradient; T is the temperature; T_0 is the temperature of the tube wall; λ is the coefficient of thermal conductivity of the fluid; J is the mechanical equivalent of heat.

Integration of the first equation of system (4) gives

$$\tau = \frac{br}{2} + \frac{C}{r}, \tag{6}$$

where C is an integration constant. From (2) and (5) it follows that $\tau = 0$ at $r = 0$. Consequently, $C = 0$.

Taking (2) and (3) into account, we reduce the system of equations (4) and (6) to the dimensionless form

$$\left|\frac{dw}{d\xi}\right|^n = \xi e^{n\theta}, \qquad \frac{d^2\theta}{d\xi^2} + \frac{1}{\xi}\frac{d\theta}{\xi} + \chi e^{-n\theta}\left|\frac{dw}{d\xi}\right|^{n+1} = 0, \tag{7}$$

where the following notation has been introduced for the dimensionless variables and parameters:

$$\xi = \frac{r}{r_0}, \qquad \theta = \frac{U}{nRT_0^2}(T - T_0), \qquad w = \left(\frac{2k_0}{br_0^{n+1}}\right)^{1/n} \exp\left(-\frac{U}{nRT_0}\right) v,$$

$$\chi = \frac{bUr_0^3}{2\lambda JnRT_0^2}\left(\frac{br_0}{2k_0}\right)^{1/n} \exp\left(-\frac{U}{RT_0 n}\right). \tag{8}$$

The exponential in expression (3) was transformed according to Frank-Kamenetskii [4], which is permissible because $nRT_0/U \ll 1$ for liquids with a strong dependence of viscosity on temperature.

The boundary conditions (5) take the form

$$\text{at } \xi = 0 \quad d\theta/d\xi = 0; \qquad \text{at } \xi = 1 \quad w = 0,\ \theta = 0. \tag{9}$$

Eliminating the velocity derivative from system (7), we obtain

$$\frac{d^2\theta}{d\xi^2} + \frac{1}{\xi}\frac{d\theta}{d\xi} + \chi\xi^{(n+1)/n}e^{\theta} = 0.$$

By the substitution $\theta = u - \frac{n+1}{n}\ln\xi$, this equation is brought to the form

$$\frac{d^2u}{d\xi^2} + \frac{1}{\xi}\frac{du}{d\xi} + \chi e^{u} = 0. \tag{10}$$

The equation obtained coincides with the equation from the theory of thermal explosion [4]. It is known that it has a solution not for all values of the parameter χ, but only for $\chi < \chi_{\text{cr}}$, where χ_{cr} is a certain critical value. For $\chi > \chi_{\text{cr}}$, equation (10) has no solution; consequently, a stationary distribution of temperatures and velocities is impossible. The heat released as a result of friction cannot be removed through the tube walls in time and leads to a progressive increase in temperature, to a hydrodynamic thermal "explosion."

The solution of equation (10) can be written in the form [5]

$$u = \ln\frac{8-c}{\chi} - 2\ln\left(a\xi^{k_1} + \frac{1}{a}\xi^{k_2}\right),$$

where a and c are integration constants; k_1 and k_2 are the roots of the equation

$$k^2 - 2k + c/8 = 0. \tag{11}$$

Passing to the function θ, we have

$$\theta = \ln\frac{8-c}{\chi} - 2\ln\left(a\xi^{k_1+(n+1)/2n} + \frac{1}{a}\xi^{k_2+(n+1)/2n}\right). \tag{12}$$

Satisfying the boundary conditions (9), we obtain

$$k_2 = -(n+1)/2n, \qquad (8-c)/\chi = (a+1/a)^2.$$

Knowing k_2, from (11) we find

$$k_1 = (5n+1)/2n, \qquad c = -2(5n+1)(n+1)/n^2.$$

To determine the constant a, a quadratic equation was obtained with roots

$$a_1 = \frac{3n+1}{2n}\sqrt{\frac{2}{\chi}} + \sqrt{\frac{2}{\chi}\left(\frac{3n+1}{2n}\right)^2 - 1},$$

$$a_2 = \frac{3n+1}{2n}\sqrt{\frac{2}{\chi}} - \sqrt{\frac{2}{\chi}\left(\frac{3n+1}{2n}\right)^2 - 1}.$$

These two roots correspond to two temperature profiles. From the stationary theory of thermal explosion it is known that one of the solutions is unstable (in the present case the stable solution corresponds to the root a_2, which henceforth we shall denote by a).

For brevity of notation, introduce the designation $\alpha = 3 + 1/n$. Then formula (12) for the temperature profile and the expression for a, taking into account the constants found, may be written in the form

$$\theta = \ln\frac{2a^2}{\chi} - 2\ln\left(a\xi^\alpha + \frac{1}{a}\right), \qquad a = \alpha\sqrt{\frac{1}{2\chi}} - \sqrt{\frac{\alpha^2}{2\chi} - 1}. \tag{13}$$

From the dependence of χ on the integration constant a,

$$\chi = \frac{2\alpha^2}{(a+1/a)^2},$$

it is seen that χ is an even function of a; as a varies from 0 to $+\infty$, χ increases from 0 to a certain maximum value and then decreases to 0. The critical value of the parameter at which the stationary flow regime is disrupted is determined from the condition $d\chi/da = 0$. From this condition it follows that $a_{\text{cr}} = 1$, $\chi_{\text{cr}} = \alpha^2/2$. As χ increases from 0 to χ_{cr}, the root a_2, corresponding to the stable solution, increases from 0 to 1, while the root a_1, corresponding to the unstable solution, decreases from ∞ to 1. Thus, for all $\chi < \chi_{\text{cr}}$ we have $a < 1$.

Let us determine the velocity field. Taking (13) and the sign of the velocity derivative into account, the first equation of system (7) can be written in the form

$$\frac{dw}{d\xi} = -\frac{2\alpha^2}{\chi}\,\frac{\xi^{\alpha-3}}{(a\xi^\alpha + 1/a)^2}.$$

After simple transformations we obtain

$$dw = -\frac{2\alpha}{\chi}\left[d\left(\frac{\xi^{\alpha-2}}{\xi^\alpha + 1/a^2}\right) + 2a^2\frac{\xi^{\alpha-3}}{1 + a^2\xi^\alpha}\right]d\xi. \tag{14}$$

The second term in (14) is not integrable for arbitrary α. However, taking into account that $a < 1$, $\xi \leq 1$, the fraction can be expanded into a uniformly convergent series and termwise integration can be performed. Integrating (14) and satisfying condition (9), we obtain the following expression for the dimensionless velocity

$$v = \frac{2r_0\alpha a^2}{\chi}\left(\frac{br_0}{2k(T_0)}\right)^{1/n}\left[\frac{1}{1+a^2} - \frac{\xi^{\alpha-2}}{1+a^2\xi^\alpha} + 2\sum_{m=0}^{\infty}\frac{(-1)^m a^{2m}}{\alpha-2+m}\left(1-\xi^{\alpha-2+m}\right)\right]. \tag{15}$$

The volume flow rate of the liquid is expressed by the formula

$$Q = \frac{4\pi r_0^3\alpha a^2}{\chi}\left(\frac{br_0}{2k(T_0)}\right)^{1/n}\left[\frac{1}{2(1+a^2)} - \frac{1}{a^2\alpha}\ln(1+a^2) + \sum_{m=0}^{\infty}\frac{(-1)^m a^{2m}}{\alpha+m}\right]. \tag{16}$$

From the solution found one can obtain the solution of the problem of isothermal flow of a non-Newtonian liquid in a cylindrical tube by taking the limiting transition as $\lambda \to \infty$. Indeed, in this case all the heat released due to internal friction will be instantaneously removed through the tube walls and will not lead to an increase in temperature. From (8) and (13) it is seen that as $\lambda \to \infty$ we have $\chi \to 0$, $a \to \sqrt{\chi}/(\alpha\sqrt{2})$. -

performing in formulas (15) and (16) the limiting transition as $\chi \to 0$, we obtain

$$v_{\text{is}} = \frac{r_0}{\alpha-2}\left(\frac{br_0}{2k(T_0)}\right)^{1/n}\left(1-\xi^{\alpha-2}\right), \qquad Q_{\text{is}} = \frac{\pi r_0^3}{\alpha}\left(\frac{br_0}{2k(T_0)}\right)^{1/n}. \tag{17}$$

For $n = 1$, the rheological equation (1) expresses the usual relation between the shear stress and the velocity gradient for a Newtonian fluid, with $k = \mu$ representing the coefficient of dynamic viscosity. Taking into account that $\alpha = 4$ for $n = 1$, from (17) we obtain formulas for the velocity profile and the discharge in the case of isothermal flow of a Newtonian fluid

$$v_{\text{is}} = \frac{br_0^2}{4\mu(T_0)}(1-\xi^2), \qquad Q_{\text{is}} = \frac{\pi b r_0^4}{8\mu(T_0)},$$

which coincide with the well-known Poiseuille formulas ([6]).

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Note: Figure translations are in progress. See original paper for figures.