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SEVERAL REMARKS ON THE METHOD OF MULTIPLIERS

MATHEMATICS

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Abstract

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MATHEMATICS

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SEVERAL REMARKS ON THE METHOD OF MULTIPLIERS

(Presented by Academician S. N. Bernstein, 9 VI 1965)

I. Let $\Lambda = \{\lambda_k^{(n)}\}$ $(k = 0, 1, \dots, n;\ n = 1, 2, \dots)$ be a triangular matrix of real numbers, with

$$\Delta_2\lambda_k^{(n)} = \lambda_k^{(n)} - 2\lambda_{k+1}^{(n)} + \lambda_{k+2}^{(n)} \quad (k = 0, 1, \dots, n-1); \qquad \lambda_0^{(n)} = 1,\ \lambda_{n+1}^{(n)} = 0. \tag{1}$$

In the theory of linear methods of summation of Fourier series, an important role is played by estimates for the mean value of the kernel, i.e. for the integral

$$\frac{1}{\pi}\int_0^\pi |K_n(t)|\,dt, \qquad K_n(t) = \frac{\lambda_0^{(n)}}{2} + \sum_{k=1}^{n} \lambda_k^{(n)} \cos kt. \tag{2}$$

A simple upper estimate is given by

Theorem 1. *If for each value of* n *the sequence of numbers* $\{\lambda_k^{(n)}\}_0^n$ *is convex or concave, i.e.*

$$\varepsilon_n \Delta_2 \lambda_k^{(n)} \geqslant 0 \quad (k = 0, 1, \dots, n-1;\ n = 1, 2, \dots), \qquad \varepsilon_n = \pm 1, \tag{3}$$

then the estimate holds

$$\frac{1}{\pi}\int_0^\pi |K_n(t)|\,dt \leqslant \frac{1}{2} + C(n+1)|\lambda_n^{(n)}|; \tag{4}$$

in the particular case where $\varepsilon_n = 1,\ \lambda_n^{(n)} \geqslant 0$, *one may take* $C = 0$.

For the proof we apply Abel' s transformation twice to the series $K_n(t)$; we obtain

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$$K_n(t) = \sum_{k=0}^{n-1} \Delta_2 \lambda_k^{(n)} \cdot S_k(t) + \lambda_n^{(n)} S_n(t), \qquad S_k(t) = \frac{1}{2}\left(\frac{\sin\frac{k+1}{2}t}{\sin\frac{1}{2}t}\right)^2,$$

$$\frac{1}{\pi}\int_0^\pi |K_n(t)|\,dt \leqslant \frac{1}{2}\sum_{k=0}^{n-1}(k+1)|\Delta_2\lambda_k^{(n)}| + \frac{1}{2}(n+1)|\lambda_n^{(n)}|;$$

from the obvious equality

$$\lambda_0^{(n)} = \sum_{k=0}^{n-1}(k+1)\Delta_2\lambda_k^{(n)} + (n+1)\lambda_n^{(n)}$$

we obtain

$$\frac{1}{\pi}\int_0^\pi |K_n(t)|\,dt \leqslant \frac{\varepsilon_n\lambda_0^{(n)}}{2} + \frac{n+1}{2}\{|\lambda_n^{(n)}| - \varepsilon_n\lambda_n^{(n)}\},$$

whence (4) follows.

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* L. Fejér ([5]) showed that in this case the kernel is nonnegative.

II. From Theorem 1 it follows: under condition (3), the additional condition

$$|\lambda_n^{(n)}| = O(1/n) \tag{5}$$

is sufficient for the validity of the inequality

$$\frac{1}{\pi}\int_0^\pi |K_n(t)|\,dt \leqslant C_1, \tag{6}$$

where C_1 does not depend on n.

Although the simple condition (5) is not necessary, it is nevertheless applicable in almost all known particular cases.

1) The Cesàro method (C, r), $r \geqslant 0$; in this case we have

$$\lambda_n^{(n)} = n!\Gamma(r+1)/\Gamma(n+r+1) \simeq e^r\Gamma(r+1)/n^r,$$

i.e., condition (5) is applicable for $r \geqslant 1$.

2) The Vallée-Poussin method

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$$\lambda_k^{(n)} = \begin{cases} 1 & (k = 0, 1, \dots, n-p), \\ (n-k+1)/(p+1) & (k = n-p+1, \dots, n); \end{cases}$$

we have $\lambda_n^{(n)} = 1/(p+1)$, and (5) is equivalent to the condition $\liminf\limits_{n\to\infty} p/n > 0$.

3) Let $\lambda_k^{(n)} = \varphi(k/(n+1))$ $(k = 0, 1, \dots, n)$, where $\varphi(0) = 1$, $\varphi(1) = 0$; the function $\varphi(x)$ has on the interval $[0,1]$ a piecewise continuous second derivative that does not change sign on this interval; then

$$\lambda_{n+1}^{(n)} - \lambda_n^{(n)} = -\lambda_n^{(n)} = \varphi(1) - \varphi\left(\frac{n}{n+1}\right) = \frac{1}{n+1}\varphi'(z),$$

$$\frac{n}{n+1} < z < 1, \qquad |\lambda_n^{(n)}| \leqslant \frac{|\varphi'(z)|}{n+1},$$

i.e., condition (5) is fulfilled owing to the boundedness of $|\varphi'(x)|$ in a neighborhood of the point $x = 1$. In particular, it is fulfilled in the method of S. N. Bernstein, where $\varphi(x) = \cos \pi x/2$, and also in the case $\varphi(x) = (1-x)^\alpha$, if $\alpha \geqslant 1$, etc.

III. Let $f(\theta) \in \mathcal{L}(-\pi, \pi)$,

$$f(\theta) \sim \sum_{k=0}^{\infty} (a_k \cos k\theta + b_k \sin k\theta), \tag{7}$$

$$U_n(f, \Lambda; \theta) = \frac{\lambda_0^{(n)} a_0}{2} + \sum_{k=1}^{n} \lambda_k^{(n)} (a_k \cos k\theta + b_k \sin k\theta).$$

Theorem 2. For the convergence

$$\lim_{n\to\infty} U_n(f, \Lambda; \theta) = f(\theta) \tag{8}$$

at every Lebesgue point of the function $f(\theta)$, the conditions (3), (5), and the additional condition

$$\lim_{n\to\infty} \lambda_k^{(n)} = 1 \qquad (k = 0, 1, \dots) \tag{9}$$

are sufficient.

The proof follows from the results of S. M. Nikol' skii ([3]).

IV. S. B. Stechkin ([4]) posed the question of finding the best constant in the inequality

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$$\left|\sum_{k=0}^{n} \frac{\lambda_k^{(n)}}{n-k+1}\right| \leqslant C \int_0^{\pi} |K_n(t)|\, dt. \tag{10}$$

Using our results ([2]) and the results of N. I. Akhiezer and M. G. Krein ([1]), one can give an algorithm for solving this problem.

Theorem 3. *The best value of the constant C in (10) is equal to the largest positive root of the equation*

$$\begin{vmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_n \\ \gamma_{-1} & \gamma_0 & \cdots & \gamma_{n-1} \\ \cdot & \cdot & \cdot & \cdot \\ \gamma_{-n} & \gamma_{-n+1} & \cdots & \gamma_0 \end{vmatrix} = 0, \qquad \gamma_{-k} = \overline{\gamma_k}, \quad \gamma_0 = \gamma + \overline{\gamma}, \tag{11}$$

where $\gamma, \{\gamma_k\}_1^n$ are the first coefficients of the expansion

$$\gamma + \sum_{k=1}^{n} \gamma_k z^k + O(z^{n+1}) = \exp\left\{\frac{i}{C}\sum_{k=0}^{n}\frac{z^k}{n-k+1} + O(z^{n+1})\right\}. \tag{12}$$

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Note: Figure translations are in progress. See original paper for figures.

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