ANALYSIS OF QUASICLASSICAL EQUATIONS

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Abstract

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ANALYSIS OF QUASICLASSICAL EQUATIONS

FOR A TWO-LEVEL SYSTEM

WITHOUT TAKING RELAXATION EFFECTS INTO ACCOUNT

(Presented by Academician I. V. Obreimov on 25 V 1965)

As is known ([1−5]), the system of nonlinear differential equations describing the interaction of matter and field is sufficiently complicated, and its solution presents great difficulties. In many cases, for simplification, a preliminary linearization of these equations is carried out. Such an approximation excludes from consideration certain essential features of the behavior of the system under study.

As is shown in the present paper, it turns out to be possible to carry out an analysis of the nonlinear equations for a two-level system while neglecting the terms describing relaxation processes. An analogous problem was solved by other methods and in another approximation in work ([6]). The treatment given in the present paper makes it possible to obtain certain new results.

For the quantities characterizing the matter and the field, one can write the following system of equations:

$$\ddot{\mathbf{P}} + \omega^2 \mathbf{P} = -k^2 W \mathbf{E}, \qquad \dot{W} = \dot{\mathbf{P}}\mathbf{E}, \qquad \ddot{\mathbf{E}} + \omega^2 \mathbf{E} = -4\pi \ddot{\mathbf{P}}. \tag{1}$$

Here $\mathbf{P}$ and W are, respectively, the polarization vector of the medium and the mean energy of the molecules per unit volume, which can be written in the form

$$\mathbf{P} = N\vec{\mu}(ab^* + a^*b), \qquad W = \frac{N\hbar\omega}{2}(bb^* - aa^*), \tag{2}$$

where N is the number of molecules per unit volume; $\vec{\mu}$ is the matrix element of the dipole-moment operator corresponding to the transition of a molecule from one energy level to another; $\hbar\omega = U_2 - U_1$ (U_2 and U_1 are the energies of the upper and lower levels of the molecule); a and b characterize the probabilities of the unexcited and excited states of the molecule (the wave function of the molecule is written in the form $\psi(t) = a(t)\psi_1 + b(t)\psi_2$).

The quantity $\mathbf{E}$ is the electric-field strength vector, and $k = 2\mu/\hbar$. To solve the system of equations (1), we prescribe the initial conditions in the convenient form $a(0) = \sin\theta$ and $b(0) = \cos\theta$. In this case, in accordance with formulas (2), the initial conditions are also determined for the quantities $\mathbf{P}(0)$, $\dot{\mathbf{P}}(0)$, and $W(0)$. We prescribe the initial conditions for the field in the form

$$\mathbf{E}(0) = \mathbf{E}_0, \qquad \dot{\mathbf{E}}(0) = 0.$$

As a result of certain transformations of the system of equations (1), one can obtain

$$\dot{P}^2 + \omega^2 P^2 + k^2 W^2 = \text{const}, \tag{3}$$

$$W + \frac{E^2}{8\pi} + \frac{1}{8\pi}\frac{(\dot{E} + 4\pi\dot{P})^2}{\omega^2} = \text{const}. \tag{4}$$

The first of these expressions can be interpreted as an implicit form of the principle of conservation of probabilities $|a|^2 + |b|^2 = 1$, and the second as a generalized law of conservation of the energy of the matter and the field.

Eliminating the quantity $\dot{P}$ from equation (4), we obtain

$$W(0) + \frac{E_0^2}{8\pi} = W + \frac{E^2}{8\pi} + \frac{\omega^2}{8\pi}\left(\int_0^t E\,dt\right)^2. \tag{5}$$

Making the substitution

$$\frac{\omega}{\sqrt{8\pi}}\int_0^t E\,dt = \omega y,$$

we reduce equation (5) to the form

$$y'^2 + \omega^2 y^2 = W(0) + E_0^2/8\pi - W, \qquad y(0) = 0. \tag{6}$$

This equation determines the time dependence of the electric-field intensity for a known function $W(t)$. Equation (6) admits numerical integration.

To determine the function $W(t)$, we shall use the well-known method of reducing second-order equations to a system of first-order differential equations. Instead of the functions $P(t)$ and $E(t)$, by means of the relations

$$P(t) = p(t)e^{i\omega t} + p^*(t)e^{-i\omega t},$$
$$E(t) = \varepsilon(t)e^{i\omega t} + \varepsilon^*(t)e^{-i\omega t} \tag{7}$$

we introduce four new functions $p, p^*, \varepsilon, \varepsilon^*$, connected by two additional relations

$$\dot{p}e^{i\omega t} + \dot{p}^* e^{-i\omega t} = 0, \qquad \dot{\varepsilon}e^{i\omega t} + \dot{\varepsilon}^* e^{-i\omega t} = 0. \tag{8}$$

In this case the system of equations (1) is brought to the form

$$\dot{p} = -\frac{ik^2 W}{2\omega}\,\varepsilon, \qquad \dot{\varepsilon} = -4\pi\dot{p} - 2\pi i\omega p,$$

$$\dot{p}^* = -\frac{ik^2 W}{2\omega}\,\varepsilon^*, \qquad \dot{\varepsilon}^* = -4\pi\dot{p}^* + 2\pi i\omega p^*, \tag{9}$$

$$\dot{W} = i\omega(p\varepsilon^* - p^*\varepsilon) + i\omega(\varepsilon p e^{2i\omega t} - \varepsilon^* p^* e^{-2i\omega t}).$$

Since what is of interest is the function $W(t)$ averaged over a time considerably longer than the period of the light oscillations, the second rapidly oscillating term in the last equation is not taken into account below.

From equations (9) one can obtain:

$$pp^* + \frac{k^2 W^2}{4\omega^2} = \text{const}, \tag{10}$$

$$\varepsilon\varepsilon^* + 2\pi W = \text{const}, \tag{11}$$

$$-\ddot{W} = 4\pi\omega^2 pp^* + k^2 W \cdot 2\pi(\varepsilon p^* + \varepsilon^* p) + k^2 W \varepsilon\varepsilon^*. \tag{12}$$

Let us estimate the conditions under which the second term on the right-hand side of the last equation may be neglected in comparison with the others. To this end, using equations (9), one may write

$$\Omega_p \sim \frac{k^2 W}{2\omega}\,\varepsilon,$$

where Ω is the frequency of variation of the quantity p. Therefore

$$\frac{2\pi\varepsilon p^* k^2 W}{4\pi\omega^2 pp^*} = \frac{\Omega}{\omega};$$

similarly,

$$\frac{2\pi p^*\varepsilon}{\varepsilon\varepsilon^*} \sim \frac{\gamma\omega}{\Omega},$$

where $\gamma = \pi N\mu^2/\hbar\omega$. Taking $N \sim 10^{15}$, $\hbar\omega \approx 10^{-13}$, $\mu \sim 10^{-18}$, we find $\gamma \sim 10^{-8}$. Assuming that the condition

$$\gamma \ll \Omega/\omega \ll 1, \tag{13}$$

is satisfied,

equation (12) can be written in the form

$$-\ddot{W} = 4\pi\omega^2 pp^* + k^2 W\varepsilon\varepsilon^*. \tag{12'}$$

In accordance with the previously chosen initial conditions for the field and the state of the molecules, we determine the initial conditions for the quantities entering the system of equations (10), (11), and (12′):

$$p(0)p^*(0) = \frac{N\mu^2}{4}\sin^2 2\theta, \qquad \varepsilon(0)\varepsilon^*(0) = E_0^2/4, \qquad \dot{W}(0) = 0. \tag{14}$$

Taking these initial conditions into account and making the change of variables

$$u = \frac{\gamma}{2}\left[\frac{W}{N\hbar\omega} + \frac{\cos 2\theta}{6}(1+\delta)\right], \qquad \tau = \omega t,$$

we obtain the canonical form of the Weierstrass equation (7)

$$\left(\frac{du}{d\tau}\right)^2 = 4u^3 - 4\gamma^2\left[1 + \frac{\cos^2 2\theta}{3}(1+\delta)^2\right]u - \frac{4\gamma^3\cos 2\theta}{27}\left[\cos^2 2\theta(2 - 21\delta + 6\delta^2 + 2\delta^3) - 9(2-\delta)\right] \tag{15}$$

with the initial condition

$$U(0) = \frac{\cos 2\theta(2-\delta)}{3}\gamma,$$

where $\delta = E_0^2/8\pi W(0)$ determines, at the initial instant of time, the ratio of the field energy to the energy of the molecules.

The solution of this equation makes it possible to analyze the character of the variation of the quantity W with time under various specific initial conditions. Let us consider some of them.

1. For $\theta = 0$, solving equation (15), one can obtain

$$W = \frac{N\hbar\omega}{2}\left(1 - \frac{\delta\ \mathrm{sn}^2\, x}{\delta/2 + 1 - \mathrm{sn}^2\, x}\right), \tag{16}$$

where $x = \tau\sqrt{\gamma(2+\delta)}$, and $\mathrm{sn}\, x$ is the Jacobi elliptic function.

From expression (16) it is seen that the quantity W varies periodically from $-N\hbar\omega/2$ to $N\hbar\omega/2$. The character of this variation depends on the magnitude of δ. For $\delta \gg 1$, the function defined by expression (16) has period π in x. Accordingly, the oscillation frequency of the quantity W is determined by the formula

$$\Omega = \omega\sqrt{\gamma\delta}. \tag{17}$$

We note that, for $\gamma \sim 10^{-8}$, relation (17) ensures fulfillment of the condition $\Omega/\omega \ll 1$ up to values $\delta \sim 10^6$.

Since the applicability of equation (12′) is also limited by the condition $\Omega/\omega \gg \gamma$, extension of solution (16) to the case of arbitrarily small δ is of no interest. Indeed, as $\delta \to 0$, as follows from solution (16), $\Omega/\omega \to 0$, and condition (13) may be violated.

2. For $\theta = \pi/2$, the solution of equation (15) has different forms for $\delta \leq 2$ and $\delta > 2$.

In the case $\delta > 2$,

$$W = -\frac{N\hbar\omega}{2}(1 - 2\,\mathrm{sn}^2\, x), \tag{18}$$

where $x = \tau\sqrt{\gamma\delta}$. In this case the quantity W varies periodically from $-N\hbar\omega/2$ to $N\hbar\omega/2$, independently of the magnitude of δ. As $\delta \to \infty$, the oscillation frequency of the quantity W is also determined by formula (17).

In the case $\delta < 2$,

$$W = -\frac{N\hbar\omega}{2}(1 - \delta\ \mathrm{sn}^2\, x), \tag{19}$$

where $x = \tau\sqrt{2\gamma}$.

In this case the quantity W varies periodically from $-N\hbar\omega/2$ to $-\frac{N\hbar\omega}{2}(1-\delta)$. As $\delta \to 0$, for the frequency of oscillation of the quantity one can obtain the expression

$$\Omega = \omega\sqrt{2\gamma}. \tag{20}$$

Hence, in the case under consideration, for $\gamma \sim 10^{-8}$, $\Omega/\omega \sim 10^{-4}$.

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Note: Figure translations are in progress. See original paper for figures.