APPLICATION OF THE CHAPLYGIN METHOD

N. F. Morozov, L. S. Srubshchik

1966

SovietRxiv

Abstract

Full Text

UDC 517.912.2

APPLICATION OF THE CHAPLYGIN METHOD TO THE STUDY OF THE MEMBRANE EQUATION

N. F. Morozov, L. S. Srubshchik

The study of the problem of the equilibrium of a membrane in the absence of chain forces on the contour reduces to the following equation [1]*):

$$u'' = -\frac{x^2}{32u^2} \tag{1}$$

with boundary conditions

$$u(0) = u(1) = 0. \tag{2}$$

To prove the existence of a solution of problem (1), (2), we shall apply Chaplygin' s method [2-4]. At the same time we shall obtain an effective method for constructing the solution.

Introduce the operator

$$L(u) \equiv u'' - \frac{x^2}{32u^2}.$$

Following the terminology of [2], we shall call $v(x)$ a lower function if $L(v) = \alpha(x) \geq 0$, and $w(x)$ an upper function if $L(w) \equiv \beta(x) \leq 0$.

We note [2] that every upper function is greater than every lower one. The following assertions are readily verified.

1. The solution of problem (1), (2) is unique in the class C_2 [4].
2. If $u(x)$ is a solution of problem (1), (2), then the inequalities

$$v(x) < u(x) < w(x) \tag{3}$$

hold.

3. As $v(x)$ and $w(x)$ one may take the functions

$$v(x) = -\frac{\sqrt[3]{9}}{4}\, x(1-x)^{\frac{2}{3}}, \qquad w(x) = -\frac{\sqrt[3]{9}}{4}\,(1-x)^{\frac{2}{3}}\left[1-(1-x)^{\frac{1}{3}}\right], \tag{4}$$

then, applying (3), we obtain

$$-\frac{\sqrt[3]{9}}{4}\, x(1-x)^{\frac{2}{3}} < u(x) < -\frac{\sqrt[3]{9}}{4}\,(1-x)^{\frac{2}{3}}\left[1-(1-x)^{\frac{1}{3}}\right]. \tag{5}$$

We shall approximate the desired solution from above. For this purpose, as the zero-th approximation w_0 we take the upper function w, and the subsequent approximations we shall seek in the form $w_n = w_{n-1} - \delta_n$, where δ_n satisfies the equation

$$\delta_n'' - \frac{M}{x(1-x)^2}\,\delta_n = \beta_{n-1}(x), \qquad \beta_{n-1} = L(w_{n-1}), \quad (n = 1,\, 2,\, 3\ldots) \tag{6}$$

with boundary conditions

$$\delta_n(0) = \delta_n(1) = 0. \tag{7}$$

It is not difficult to establish that if $\beta_{n-1}(x) \leq 0$, then $\delta_n(x)$ is nonnegative. Now we choose the constant M from the condition

$$\beta_n(x) = \frac{x^2}{32}\left[\frac{1}{w_{n-1}^2} - \frac{1}{(w_{n-1}-\delta_n)^2}\right] - \frac{M\delta_n}{x(1-x)^2} \leq 0. \tag{8}$$

Taking into account that $\delta_n(x) \geqslant 0$, we successively derive

$$-\frac{M\delta_n}{x(1-x)^2} - \frac{x^2(2w_{n-1}-\delta_n)(-\delta_n)}{32w_{n-1}^2(w_{n-1}-\delta_n)^2} \geqslant \delta_n\left[-\frac{M}{x(1-x)^2} - \frac{x^2}{16|w_{n-1}|^3}\right] \geqslant$$

$$\geqslant \delta_n\left[-\frac{M}{x(1-x)^2} - \frac{4}{9}\,\frac{x^2}{(1-x)^2\left[1-(1-x)^{1/3}\right]^3}\right] =$$

$$= \frac{\delta_n}{x(1-x)^2}\left\{M - \frac{4}{9}\left[\frac{x}{1-(1-x)^{1/3}}\right]^3\right\} \geqslant 0.$$

Hence it follows that it is sufficient to take $M = 12$ in order that (8) be satisfied for all natural n.

---

*) The present paper corrects an inaccuracy made in [1].

By direct verification one can ascertain that

$$y_1 = \frac{x^3}{(1-x)^3} + \frac{x^2}{(1-x)^2} + \frac{x}{5(1-x)}$$

is a solution of the homogeneous equation (6). Then the second linearly independent solution of the homogeneous equation (6) is equal to

$$y_2 = y_1 \int_x^1 \frac{(1-x_1)^6}{\left[x_1^3 + x_1^2(1-x) + \frac{1}{5}x_1(1-x_1)^2\right]^2}\, dx_1,$$

and the solution of the nonhomogeneous equation (6) under the boundary conditions (7) has the form

$$\delta_n = -\int_x^1 \beta_{n-1}(t) y_2(t)\, dt \cdot y_1(x) - \int_0^x \beta_{n-1}(t) y_1(t)\, dt \cdot y_2(x). \tag{9}$$

Let us now consider the process of successive approximations

$$w_n = w_{n-1} - \delta_n \tag{10}$$

or

$$w_n = w_0 - (\delta_1 + \delta_2 + \cdots + \delta_n).$$

All δ_n are found from equation (6) with $M = 12$, and the right-hand sides $\beta_{n-1}(x)$ are found from the relation

$$\beta_{n-1}(x) = L(w_{n-1}).$$

For all n, $\beta_n(x)$ have the form $\beta_n(x) = \gamma_n(x)(1-x)^{-4/3}$, where $\gamma_n(x)$ are continuous functions on $[0,1]$. Further, from (9) one can establish that all $\delta_n(x) = \varepsilon_n(x)(1-x)^{2/3}x$, where $\varepsilon_n(x)$ are continuous functions on $[0,1]$,

$$w_0 \geqslant w_1 \geqslant w_2 \geqslant \cdots \geqslant w_n \geqslant \cdots \geqslant v. \tag{11}$$

Hence it follows that for all $x \in [0,1]$, $w_n \to u_0$, i.e.

$$u_0 = w_0 - \sum_{n=1}^{\infty} \delta_n = w_0 - x(1-x)^{2/3} \sum_{n=1}^{\infty} \varepsilon_n.$$

Then from (4) we obtain the inequality

$$\sum_{n=1}^{\infty} \varepsilon_n \leqslant \frac{\sqrt[3]{9}}{4}. \tag{12}$$

Consequently, $\varepsilon_n \to 0$ as $n \to \infty$.

Finally, let us prove that $u_0(x)$ is a solution of problem (1), (2). For this it is sufficient to show that $u_0(x)$ satisfies the integral equation equivalent to (1), (2),

$$u_0(x) = \int_0^x dt \int_0^t \frac{\xi^2\, d\xi}{32\, u_0^2} - x \int_0^1 dt \int_0^t \frac{\xi^2\, d\xi}{32\, u_0^2}.$$

Consider the equality

$$L(w_n) = \beta_n(x). \tag{13}$$

Integrating the identity (13) twice, taking into account the boundary conditions (2), we obtain

$$\begin{aligned} w_n(x) = \int_0^x dt \int_0^t \frac{\xi^2\, d\xi}{32\, w_n^2} - x \int_0^1 dt \int_0^t \frac{\xi^2\, d\xi}{32\, w_n^2} + \\ + \int_0^x dt \int_0^t \beta_n(\xi)\, d\xi - x \int_0^1 dt \int_0^t \beta_n(\xi)\, d\xi. \end{aligned}$$

As is seen from (8),

$$|\beta_n(x)| < \frac{M\, \delta_n(x)}{x(1-x)^2} = \frac{M\, \varepsilon_n(x)}{(1-x)^{4/3}}. \tag{14}$$

Letting n tend to infinity and applying Lebesgue' s theorem (see [5], p. 134), we obtain in the limit

$$u_0(x) = \int_0^x dt \int_0^t \frac{\xi^2\, d\xi}{32\, u_0^2(\xi)} - x \int_0^1 dt \int_0^t \frac{\xi^2\, d\xi}{32\, u_0^2(\xi)},$$

which was required to be proved.

References

1. Morozov N. F., *Dokl. Akad. Nauk SSSR*, **152**, No. 1, 1963.
2. Babkin B. N., *PMM*, **18**, issue 2, 1954.
3. Srubshchik L. S., Yudovich V. I., *PMM*, **26**, issue 5, 1962.
4. Srubshchik L. S., Yudovich V. I., *Siberian Mathematical Journal*, **4**, No. 3, 1963.
5. Natanson I. P., *Theory of Functions of a Real Variable.* Moscow–Leningrad, GITTL, 1950.

Received by the editors
May 25, 1965

Leningrad Technological Institute
of the Pulp and Paper Industry

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.