# ON THE VARIATIONAL THEORY OF NONLINEAR EQUATIONS

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## Abstract

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*MATHEMATICS*

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# ON THE VARIATIONAL THEORY OF NON-LINEAR EQUATIONS

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1. The question of the existence of a solution of the equation

$$BF(x) = x, \tag{1}$$

where $B$ is a linear, indefinite operator and $F(x)$ is a potential operator, was investigated in papers ([1,2]). In doing so, however, it was assumed that $B$ is a bounded, self-adjoint operator whose positive spectrum consists of a finite number of eigenvalues of finite multiplicities.

In the present paper we establish new theorems on the existence of a solution of equation (1) without the assumption of boundedness of the operator $B$. We also do not assume that the positive part of the spectrum of the operator $B$ consists of a finite number of eigenvalues, each of finite multiplicity. This circumstance makes it possible to apply the results established here not only to nonlinear integral equations of Hammerstein type, but also to certain boundary-value problems for nonlinear differential equations.

In the second part we consider equations of the form (1) in which the operator $B$ is still unbounded, while $F(x)$ is not assumed to be potential.

Using the concept of monotonicity of operators, in a number of cases one can prove not only the existence of a solution of equation (1), but also its uniqueness.

2. In this section we shall consider the operator equation (1) in a Hilbert space $H$ under the assumption that $B$ is a linear, generally speaking unbounded operator, and $F(x)$ is a potential operator acting in $H$.

**Theorem 1.** *Suppose the following conditions are satisfied:*

$1^\circ$. *$B$ is a self-adjoint operator, defined on a dense set in $H$, whose negative spectrum is separated, and whose positive spectrum is contained in the interval $(m, +\infty)$, where $m > 0$, and is otherwise separated.*

$2^\circ$. *The operator $F(x) = \operatorname{grad} f(x)$ is weakly continuous.*

3°. *The functional* $f(x)$ *is weakly lower semicontinuous and*

$$f(x) \leq a_1(x,x) + a_2(x,x)^\gamma + a_3,$$

*where*

$$a_1 \geq 1/m, \quad \text{if } m < 1; \qquad a_1 \geq 1 + 1/m, \quad \text{if } m \geq 1,$$

$$a_2 \leq 0, \quad a_3 \leq 0, \quad 0 < \gamma < 1.$$

*Then there exists at least one element* $z_0 \in H$ *such that*

$$BF(z_0) = z_0.$$

The proof of the theorem is based on the property of weakly lower semicontinuous functionals of attaining an exact lower bound on an arbitrary bounded, weakly closed set in $H$ (see (2), p. 107, Theorem 9.2).

**Remark 1.** We assume that the dimension of the subspace $H_+$, onto which the projection operator is $E - E_m$, where $E_m$ is the value at $t = m$ of the spectral function $E_t$ of the operator $B$, is at most countable.

**Remark 2.** The stated conditions are satisfied, for example, by a Nemitskii operator of the form $a(x)+b(x)u$, where $a(x) \in L^2$, $b(x) \in L^\infty$ and $b(x) \geq \alpha > 0$, considered in the space $L^2$ (see (4)). As the operator $B$ one may take an arbitrary linear self-adjoint operator, defined on a dense set in $H$, whose positive spectrum is contained in the interval $(m, +\infty)$ with $m > 2/\alpha$.

In the following theorems we impose no restrictions on the subspace $H_+$.

**Theorem 2.** *Suppose the following conditions are satisfied:*

1°. $B$ *is a self-adjoint operator, defined on a dense set in* $H$, *with arbitrary negative spectrum.*

2°. *The operator* $B_+^{-1}$, *inverse to the positive part* $B_+$ *of the operator* $B$, *exists and is completely continuous.*

3°. *The functional* $f(x)$ *is weakly lower semicontinuous and*

$$f(x) \geq a_1(x,x) + a_2(x,x)^\gamma + a_3,$$

where $a_1 \geq \frac{1}{2}(1 + \|B_+^{-1}\|)$, $a_2 \leq 0$, $a_3 \leq 0$, $0 < \gamma < 1$.

*Then the operator equation* $BF(x) = x$ *is solvable.*

**Theorem 3.**

1°. *Let $B$ be a self-adjoint operator, defined on a dense set in $H$, whose negative spectrum is arbitrary, while its positive spectrum is contained in the interval $(m, +\infty)$, where $m > 0$, and otherwise is arbitrary.*

2°. *$F(x)$ is the gradient of some functional $f(x)$ such that at each fixed point $x_0 \in H$ the inequality holds*

$$f(x) - f(x_0) \geq (F(x_0), x - x_0) + \alpha\|x - x_0\|^2,$$

where

$$\alpha \geq 1/m, \quad \text{if } m < 1; \qquad \alpha \geq 1 + 1/m, \quad \text{if } m \geq 1.$$

*Then there exists at least one vector $z_0 \in H$ such that*

$$BF(z_0) = z_0.$$

**Theorem 4.** *Suppose the following conditions are satisfied:*

1°. *$B$ is a self-adjoint operator, defined on a dense set in $H$, whose negative spectrum is arbitrary, while its positive spectrum is contained in the interval $(m, +\infty)$, where $m > 0$, and otherwise is arbitrary.*

2°. *$F(x)$ is a potential operator acting in $H$ and satisfying, for any $x, h \in H$, the inequality*

$$(DF(x, h), h) \geq 2a_1(h, h),$$

where $a_1$ is the constant defined in the same way as in Theorem 1.

*Then the solution of the operator equation $BF(x) = x$ exists and is unique.*

**Theorem 5.**

1°. *Let $B$ be a self-adjoint operator, defined on a dense set in $H$, whose negative spectrum is arbitrary, while its positive spectrum is contained in the interval $(m, +\infty)$, where $m > 0$, and otherwise is arbitrary.*

2°. $F(x)$ is a potential operator acting in $H$ and, for any $x, h \in H$, satisfies the condition

$$(F(x + h) - F(x), h) \geq 2a_1(h, h),$$

where $a_1 \geq 1/m$ if $m < 1$; $a_1 \geq 1 + 1/m$ if $m \geq 1$.

Then the solution of the operator equation $BF(x) = x$ in $H$ exists and is unique.

3. Up to now we have restricted ourselves to the case when the operator $F(x)$ in equation (1) is potential. Here we shall consider the equation $BF(x) = x$ without assuming the potentiality of the operator $F(x)$.

**Theorem 6.**
1°. Let $B$ be a self-adjoint operator defined on a dense set in $H$, whose negative spectrum is arbitrary, and whose positive spectrum is contained in the interval $(m, +\infty)$, where $m > 0$, and is otherwise arbitrary.

2°. The operator $F(x)$, acting in $H$, satisfies the conditions:

a) $\|F(x+h) - F(x)\| \leq N(r)\|h\|$, $x$, $x+h \in D_r$, where $D_r$ is the ball of radius $r$;

b) $(h, F(x+h) - F(x)) \geq 2a_1(h,h)$, where $a_1 \geq 1/m$ if $m < 1$; $a_1 \geq 1 + 1/m$ if $m \geq 1$.

Then the solution of the operator equation $BF(x) = x$ exists and is unique.

**Theorem 7.**
1°. Let $B$ be a self-adjoint operator defined on a dense set in $H$, whose negative spectrum is arbitrary, and whose positive spectrum is contained in the interval $(m, +\infty)$, where $m > 0$, and is otherwise arbitrary.

2°. The operator $F(x)$, acting in $H$, is Gâteaux differentiable and satisfies the conditions:

a) $\|F'(x)\| \leq M$;

b) $(h, F'(x)h) \geq 2a_1(h,h)$, where $a_1$ is a constant determined in the same way as in Theorem 1.

Then the solution of the operator equation $BF(x) = x$ in $H$ exists and is unique.

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*Note: Figure translations are in progress. See original paper for figures.*