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Abstract

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MATHEMATICS

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ELLIPTIC OPERATORS IN FUNCTIONAL DERIVATIVES AND THE DIFFUSION EQUATIONS ASSOCIATED WITH THEM

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1°. To a second-order differential expression

$$l_2(F) = \sum_{j,k} b_{jk} \frac{\partial^2 F}{\partial x_j \partial x_k}$$

with respect to a function $F(x_1, \dots, x_n)$ one may give the form

$$l_2(F) = \mathrm{Sp}(BF''), \tag{1}$$

where $B = \|b_{jk}\|$ is the matrix of coefficients; $F'' = \|f_{jk}\|$ is the matrix of second derivatives $f_{jk} = \partial^2 F/\partial x_j \partial x_k$. Formula (1) can be generalized to the case when $F(x)$ is a functional on an infinite-dimensional space. For the case when the operator B is bounded, this was done in (^{1}). Here we describe a construction that makes it possible to consider also unbounded operators B.

Let us agree on some notation. Let $\mathfrak{B}$ be a Banach space; $\mathfrak{B}^*$ the space conjugate to it. The value of a functional $\xi \in \mathfrak{B}^*$ on an element $\varphi \in \mathfrak{B}$ will be denoted by the symbol (φ, ξ). A nonlinear functional $F(x)$ in $\mathfrak{B}$ will be called twice continuously differentiable (class $C_2(\mathfrak{B})$) if the representation

$$F(x+h) - F(x) = (h, F'(x)) + (h, F''(x)h) + o(\|h\|), \tag{2}$$

holds, where $F(x) \in \mathfrak{B}^*$, $F''(x) \in \{\mathfrak{B} \to \mathfrak{B}^*\}$, and these expressions are continuous functions of the argument x in the corresponding norms. By $\{\mathfrak{B}_1 \to \mathfrak{B}_2\}$ we shall always mean the space of linear bounded operators acting from $\mathfrak{B}_1$ into $\mathfrak{B}_2$.

An operator A in a Hilbert space $\mathfrak{H}$ will be called an operator of class $\mathfrak{S}_1$ if it has finite absolute trace, i.e., the series

$$\sum_{k=1}^{\infty}(A\varphi_k, \varphi_k)$$

(φ_k is a complete orthonormal system in $\mathfrak{H}$) converges absolutely, and an operator of class $\mathfrak{S}_2$ (a Hilbert-Schmidt operator) if

$$\sum_{k=1}^{\infty}\|A\varphi_k\|^2 < \infty.$$

The symbols D_A, R_A will denote, respectively, the domain of definition and the range of the operator A.

2°. Let $\mathfrak{H}_k$ $(k = 1, 2)$ be Hilbert spaces, and let D_k be a linear set dense in $\mathfrak{H}_k$. Consider a linear operator A mapping D_2 into D_1, and suppose that the following conditions are satisfied:

a) in D_k a new norm $\|x\|_k^+$ can be introduced in such a way that the operator A becomes bounded:

$$\|Ax\|_1^+ \leq C\|x\|_2^+ \quad (x \in D_2);$$

b) the relation

$$\|x\|_k^+ = \|T_{kx}\|_k \quad (x \in D_k)$$

holds, where T_k is a positive definite operator in $\mathfrak{H}_k$, with $D_{T_k} \supset D_k$, and there exists an inverse operator T_k^{-1} (possibly unbounded);

c) the operator T_1^{-1} belongs to the class $\mathfrak{S}_2$.

Let $\mathfrak{H}_k^+$ be the completion of D_k in the norm $\|\cdot\|_k^+$; let $\mathfrak{H}_k^-$ be the completion of DT_k^{-1} in the norm $\|x\|_k^- = \|T_k^{-1}x\|_k$. Denote by $\hat{T}_k, \hat{T}_k^{-1}$ the closures of the operators T_k, T_k^{-1} in the corresponding norms. Then $\hat{T}_k\mathfrak{H}_k^+ \subset \mathfrak{H}_k$, $\hat{T}_k^{-1}\mathfrak{H}_k^{-1} \subset \mathfrak{H}_k$, and the spaces $\mathfrak{H}_k^+, \mathfrak{H}_k^-$ can be interpreted as mutually dual, with

$$(\varphi, \xi) = (\hat{T}_k\varphi, \hat{T}_k^{-1}\xi)_k \qquad (\varphi \in \mathfrak{H}_k^+,\ \xi \in \mathfrak{H}_k^-).$$

From condition c) there follows the embedding $\mathfrak{H}_1^+ \subset \mathfrak{H}_1 \subset \mathfrak{H}_1^-$ and membership in the class $\mathfrak{S}_1$ of every operator $B \in \{\mathfrak{H}_1^- \to \mathfrak{H}_1^+\}$. Condition a) implies the possibility of extending the operator A to all of $\mathfrak{H}_2^+$ so that $A \in \{\mathfrak{H}_2^+ \to \mathfrak{H}_1^+\}$. In this case $A^* \in \{\mathfrak{H}_1^- \to \mathfrak{H}_2^-\}$.

Consider a nonlinear functional $F(x)$ of class $C_2(\mathfrak{H}_2^-)$. For it $F''(x) \in \{\mathfrak{H}_2^- \to \mathfrak{H}_2^+\}$, and consequently $AF''(x)A \in \{\mathfrak{H}_1^- \to \mathfrak{H}_1^+\} \subset \mathfrak{S}_1$. Thus one can introduce the second-order differential operator $l_2(F) = \operatorname{Sp}\{AF''(x)A^*\}$, which it is natural to call elliptic.

3°. Consider a diffusion equation of the form

$$\partial F/\partial t + \frac{1}{2}\operatorname{Sp}\{A(x,t)F''A^*(x,t)\} + (F', a(x,t)) = 0, \tag{3}$$

where the coefficients $a(x,t) \in \mathfrak{H}_2^-$, $A(x,t) \in \{\mathfrak{H}_2^+ \to \mathfrak{H}_2^-\}$ are sufficiently smooth. For this equation one poses the Cauchy problem on the interval $t_0 \leq t \leq t_1$ with condition

$$F(x, t_1) = \Phi(x), \tag{4}$$

where $\Phi(x)$ is a bounded functional in $\mathfrak{H}_2^-$. (In connection with the consideration of the backward diffusion equation, the initial condition is imposed at the right-hand end of the interval; by the substitution $t = -t'$ one easily passes to the usual formulation of the problem.)

The solution of problem (3)—(4), under the conditions described below, can be represented in the form of the integral

$$F(x,t) = \int_{\mathfrak{H}_2^-} \Phi(y)\, \mu(t, x; t_1; dy) \tag{5}$$

with respect to the probability measure $\mu(t, x; \tau, \cdot)$, defined on a certain σ-ring of subsets of the space $\mathfrak{H}_2^-$ containing all Borel cylinder sets. This measure depends on the parameters $t, \tau \in [t_0, t_1]$, $x \in \mathfrak{H}_2^-$, and is the probability measure generated by a random variable $\xi(\tau)$ taking values in the space $\mathfrak{H}_2^-$ and satisfying the stochastic differential equation

$$d\xi(\tau) = a(\xi, \tau)d\tau + A^*(\xi, \tau)dw(\tau), \qquad t \leq \tau, \tag{6}$$

and the condition $\xi(\tau)\big|_{\tau=t} = x$. Here $w(\tau)$ is a Wiener random process with values in $\mathfrak{H}_1^-$, i.e. a normal homogeneous random process with independent increments, for which the increment $w(\tau_2) - w(\tau_1)$ has zero mean and correlation operator $(\tau_2 - \tau_1)I$. The existence of such a process w follows from a known result of Minlos (see ([2],[3])).

Theorem. *Let the coefficients $a(\xi, \tau)$, $A(\xi, \tau)$ satisfy, in the corresponding norms, the Lipschitz condition in ξ with a constant independent of $\tau \in (t_0, t_1)$, and be continuous in τ. There exists, unique up to stochastic equivalence, a solution of equation (6) satisfying the condition $\xi(t) = x$ and continuous with probability one. This solution is a Markov process with values in $\mathfrak{H}_2^-$.*

If the functions $a(\xi, \tau)$, $A(\xi, \tau)$, $\Phi(\xi)$ belong to the class C_2 in the corresponding spaces, then formula (5) gives the solution of problem (3)—(4).

The proof of the theorem is carried out in the same way as in the finite-dimensional case (see, for example, ([4])). In doing so one uses estimates of stochasticintegrals in the space $\mathfrak{H}_1^-$, given in (1), and the Itô differential formula, which for a functional $u(t,x)$ on $\mathfrak{H}_2^-$ takes the form

$$du(t,\xi(t)) = \{u'_t(t,\xi(t)) + (u'_x(t,\xi(t)),\ a) + {}^1/2\,\mathrm{Sp}(Au''_{xx}(t,\xi(t))A^*)\}\,dt + \\ + (u'_x(t,\xi(t)), A^* dw). \tag{7}$$

4°. Let us consider several special cases.

1) Let $A(t) \in \mathfrak{S}_2(t_0 \le t \le t_1)$ and suppose there exists $\tau \in [t_0, t_1]$ for which the operator $A^{-1}(\tau)$ exists (unbounded), with $R_{A(t)} \subseteq R_{A(\tau)}$. Put $T_1 = |A(\tau)|^{-1}$, $T_2 = I$. Then $\mathfrak{H}_1^+ = \bar{R}_{A(\tau)} \subset \mathfrak{H}$; $\mathfrak{H}_2^+ = \mathfrak{H}_2^- = \mathfrak{H}$, and thus the measure μ turns out to be concentrated in $\mathfrak{H}$. The operator $B = A^*A \in \mathfrak{S}_1$ has meaning in $\mathfrak{H}$. If one takes $\mathfrak{H} = \mathcal{L}_2[a,b]$, then to this operator there corresponds a certain kernel $b(s_1,s_2)$; let $\delta^2 F/\delta x(s_1)\delta x(s_2)$ be the generalized kernel corresponding to the operator F'' (the second variational derivative of the functional $F(x)$). The operator $l_2(F)$ takes the form

$$l_2 = \int_a^b \int_a^b b(s_1,s_2) \frac{\delta^2 F}{\delta x(s_1)\delta x(s_2)}\, ds_1 ds_2.$$

Equations with such operators were in fact considered in ([5],[6]). We note that the general stochastic equation (6) can be reduced to an equation with an operator $A \in \mathfrak{S}_2$ by expanding the process $w(t)$ in the eigenvectors of the operator T (see ([1])).

The case when $A^m \in \mathfrak{S}_2$ is considered analogously. Imposing on the operator $A(t)$ more stringent requirements (for example, $A = A_1^k$, where $A_1 \in \mathfrak{S}_2$), we shall obtain equations of the form (3), for which the Cauchy problem is solvable for broader classes of functionals $\Phi(x)$ (for example, those depending on derivatives of the function $x(s)$).

2) The operator $A(t)$ is bounded in $\mathfrak{H}$ and leaves invariant some domain D_T, where $T^{-1} \in \mathfrak{S}_2$. This case was considered in ([1]). If it is known only that $A(t)$ is bounded in $\mathfrak{H}$, then one can enlarge $\mathfrak{H}$ by completing it with respect to a certain weaker norm of the form $\|x\|_- = \|T^{-1}x\|$ and take $\mathfrak{H}_1^+ = \mathfrak{H}_2^+ = \mathfrak{H}$, taking the completed space as the basic one. In this case the measure will be concentrated in an even broader space.

3) Let, generally speaking, $A(t)$ be an unbounded operator in $\mathfrak{H}$ having the property that, for some $\tau \in [t_0, t_1]$, $A^{-1}(\tau) \in \mathfrak{S}_2$, and moreover $D_{A(t)A(\tau)} \supseteq D_{A^2(\tau)}$ (this is the case, for example, if the domain $D_{A^2(t)}$ is constant except at isolated points where $A(t) = 0$). In this case one may take $T_1 = |A(\tau)|$, $T_2 = |A^2(\tau)|$, $\mathfrak{H}_1^+ = D_{A(\tau)}$, $\mathfrak{H}_2^+ = D_{A^2(\tau)}$. Cases in which the condition $A^{-\alpha}(\tau) \in \mathfrak{S}_2$ holds for some α greater or less than

one are considered analogously. In the case of the space $\mathfrak{H} = \mathcal{L}_2[a, b]$, the considerations set forth make it possible to consider equations (3) and (6) with a differential operator $A(t)$.

5°. Consider a sequence of equations of type (6), whose coefficients $a_n(\xi, t)$ and $A_n(\xi, t)$ $(n = 0, 1, 2, ...)$ satisfy the conditions of Theorem 1, and let $\xi_n(t)$ be solutions of these equations satisfying one and the same condition $\xi_n(t_0) = x$.

Theorem 2. Let, for every $x \in \mathfrak{H}_1^-$,

$$\lim_{n\to\infty} \|A_0^*(\xi,\tau)x - A_n^*(\xi,\tau)x\|_2^- = 0 \quad \text{and} \quad \lim_{n\to\infty} \|a_n(\xi,\tau) - a_0(\xi,\tau)\|_2^- = 0 \quad (\xi \in \mathfrak{H}_{-2},$$

$t_0 \leq \tau \leq t_1$), and suppose that the functions $a_n(\xi, \tau)$ and $A_n(\xi, \tau)$ have one and the same Lipschitz constant. Then

$$\lim_{n\to\infty} \left\{ \sup_t M\|\xi_0(t) - \xi_n(t)\|^2 \right\} = 0.$$

In this case

$$F_0(x, t) = \lim_{n\to\infty} F_n(x, t),$$

where F_n is the solution of the Cauchy problem (5), (3)–(4) with coefficients a_n, A_n $(n = 0, 1, ...)$.

Using this theorem, by approximating, in the strong sense, the coefficients of problem (3)–(4) by sequences of finite-dimensional operators, we can represent the solution of this problem as the limit of finite-dimensional problems. Since the solutions of these problems are unique in the class of functionals under consideration, in this way one can obtain a proof of the uniqueness theorem for the solution of problem (3)–(4).

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Note: Figure translations are in progress. See original paper for figures.