ON THE EQUILIBRIUM POTENTIAL

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Abstract

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MATHEMATICS

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ON THE EQUILIBRIUM POTENTIAL

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§ 1. Let R_n be n-dimensional Euclidean space, $n \geqslant 1$. Denote by Π an n-dimensional closed parallelepiped. Let $\mathfrak{B}$ be a system of Borel sets from Π, and let μ be a measure defined on $\mathfrak{B}$, i.e., a completely additive nonnegative set function defined on $\mathfrak{B}$. Normalization means $\mu(\Pi) = 1$. If $B \in \mathfrak{B}$ and $\mu(B) = 1$, then one says that μ is concentrated on B. This circumstance is denoted by $\mu \prec B$.

We shall say that a function $\varphi(t)$ is a kernel of the class Φ if $\varphi(t) > 0$ is a continuous decreasing function defined for $t > 0$ and such that

$$\lim_{t \to 0} \varphi(t) = +\infty \quad \text{and} \quad \int_0^1 t^{n-1} \varphi(t)\, dt < +\infty.$$

Denote by r_{PQ} the ordinary Euclidean distance between the points $P = (x_1, \dots, x_n)$ and $Q = (y_1, \dots, y_n)$ of R_n. The Lebesgue–Stieltjes integral

$$u(P) = \int_{\Pi} \varphi(r_{PQ})\, d\mu(Q) \tag{1}$$

defines at each point of R_n a function $u(P)$, called the Φ-potential generated by the measure μ. We note that all integrals encountered below will be understood in the Lebesgue–Stieltjes sense.

§ 2. Let F be a closed set from Π, and let μ be a measure concentrated on the set F. Put

$$I(\mu) = \iint_F \varphi(r_{PQ})\, d\mu(P)\, d\mu(Q), \tag{2}$$

$$W(F) = \inf_{\mu \prec F} I(\mu). \tag{3}$$

Following Frostman ([1]), the φ-capacity of a bounded closed set F is the number $C_\varphi(F)$ determined by the equation $W(F) = \varphi[C_\varphi(F)]$. For an arbitrary set E from Π, the φ-capacity is defined by putting $C_\varphi(E) = \sup_{F\subset E} C_\varphi(F)$, where F is a closed subset of E.

§ 3. Let F be a closed set from Π, and suppose that for some class of kernels there exists a measure $\mu^* \prec F$ such that $I(\mu^*) = W(F)$; then the measure μ^* is called an equilibrium measure, and the φ-potential $u^*(P)$ generated by it is called the equilibrium potential.

In the present paper we consider the problem of existence and uniqueness of the equilibrium measure for a broader class of kernels $\varphi(t)$ from Φ than in ([1,5]).

§ 4. On the existence of the equilibrium potential. Denote by F_μ the support of the measure μ (see ([3])); it is known that the set F_μ is closed. Using definition (3) and the selection theorem ([1,3]), it is not difficult to show that if $\varphi(t) \in \Phi$ and F is a closed set from Π of positive-

···of positive φ-capacity, then there exists an equilibrium measure $\mu^* \prec F$. We shall say that the kernel $\varphi(t) \in \Phi_1$, if $\varphi(t) \in \Phi$, and if for the given $n \geq 1$ there exists a number $h > 0$ such that $0 < h < n$ and $t^h\varphi(t)$ does not decrease as t increases.

Lemma 1. Let $\varphi(t) \in \Phi_1$, and let the φ-potential $u(P)$, generated by some measure μ, be bounded on the support of this measure F_μ; then $u(P)$ is bounded everywhere in R_n.

Proof. Let $P \notin F_\mu$; denote by P_0 the point of F_μ nearest to P, or one of them if there are several. Such a point P_0, at least one, exists, since F_μ is closed. Since $r_{P_0Q} \leq 2r_{PQ}$ and $\varphi(t) \in \Phi_1$, we find $u(P) \leq 2^h u(P_0)$, which proves the lemma.

Lemma 2. Let $\varphi(t) \in \Phi_1$, and let the φ-potential $u(P)$, generated by a measure μ, be continuous on the support of this measure; then it is continuous everywhere in R_n.

Proof. Denote by $S_\delta(P)$ the n-dimensional ball of radius δ with center at the point P. Put

$$u_\delta(P) = \int_{S_\delta(P)\cap F_\mu} \varphi(r_{PQ})\, d\mu(Q). \tag{4}$$

From the continuity of $u(P)$ on the closed set F_μ it follows that for any $\varepsilon > 0$ there exists such a $\delta(\varepsilon) > 0$ that $u_\delta(P) < \varepsilon$ for every point $P \in F_\mu$. If, however, $P \notin F_\mu$ and P is at distance less than δ from F_μ, then, carrying out a proof analogous to the proof of Lemma 1, we find $u_\delta(P) \leq 2^h u_{2\delta}(P_0)$, which proves the lemma.

Lemma 3. Let $\varphi(t) \in \Phi_1$, and let $u(P)$ be a φ-potential generated by some measure μ; then

$$m_r(P) \leq Au(P), \tag{5}$$

where $m_r(P)$ is the n-dimensional mean value of the φ-potential $u(P)$; A is a constant independent of P.

Proof. Substituting in $m_r(P)$ the expression for $u(P)$ from (1) and changing the order of integration, we find

$$m_r(P) = \int_{F_\mu} d\mu(M) \cdot \frac{1}{V_r} \int_{\bar{S}_r(P)} \varphi(r_{PQ})\, dv_Q = \int_{F_\mu} f(r, P, M)\, d\mu(M), \tag{6}$$

where $\bar{S}_r(P)$ is the closed n-dimensional ball; V_r is the volume of this ball; dv_Q is the volume element at the point Q. Suppose that the point $M \in \bar{S}_r(P)$; then, putting $t = r_{MQ}$ and taking into account that $r_{MQ} \leq 2r$ and $\varphi(t) \in \Phi_1$, we find

$$f(r, P, M) \leq \frac{n}{r^n} \int_0^{2r} t^{n-1} \varphi(t)\, dt \leq \frac{2^n n}{n-h} \varphi(r_{PM}). \tag{7}$$

If $r < r_{MP} \leq 2r$, then, denoting by P_0 the point of $\bar{S}_r(P)$ nearest to M and taking into account that $(\varphi(r_{MQ})/\varphi(r_{P_0Q})) \leq 2^h$ and $V_{2r}/V_r = 2^n$, we obtain $\varphi(r_{MQ}) \leq 2^h \varphi(r_{P_0Q})$ and

$$f(r, P, M) \leq 2^{h+n} \frac{n}{n-h} \varphi(r_{PM}). \tag{8}$$

If $r_{PM} > 2r$, then $r_{PM} \leq r_{PQ} + r_{QM} \leq 2r_{QM}$, and therefore $f(r, P, M) \leq 2^h \varphi(r_{PM})$. Hence, from (7) and (8), we obtain (5), where $A = 2^{h+n} n/(n-h)$. The lemma is proved.

Theorem 1. Let $\varphi(t) \in \Phi_1$, and let F be a closed set from Π of positive φ-capacity; then: 1) $W(F) \leq u^*(P)$ nearly everywhere in the sense of φ-capacity on F, and 2) $u^*(P) = W(F)$ nearly everywhere in the sense of φ-capacity on F_{μ^*}.

Proof. Since μ^* is an equilibrium measure, for arbitrary $\varepsilon > 0$ the inequality $u^*(P) \leq W(F) - \varepsilon$ cannot hold everywhere on F. Consequently, there exists a point $P_0 \in F$, for which

$u^*(P) > W(F) - \varepsilon$, and since the φ-potential is lower semicontinuous everywhere in R_n, there exists a neighborhood $O(P_0)$ such that $u^*(P) > W(F) - \varepsilon$ for all $P \in O(P_0)$. Let D be a subset of F where $u^*(P) \leq W(F) - 2\varepsilon$; the set D is closed. Suppose that $C_\varphi(D) > 0$. Define the set function ν, putting $\nu = -\mu^*$ in $O(P_0)$; $\nu > 0$ on D and $\nu(D) = \mu^*[O(P_0)] = m$; $\nu = 0$ on $D \cup O(P_0)$, and $I(D, \nu) < +\infty$.

Put $\eta = \mu^* + h_1\nu$, where $0 < h_1 < 1$. We have $\delta I = I(\mu^* + h_1\nu) - I(\mu^*) \geq 0$, since $\eta \prec F$. On the other hand, we have $\delta I < -h_1[2m\varepsilon - h_1 I(F, \nu)]$. Consequently, for sufficiently small $h_1 > 0$ we obtain $\delta I < 0$. The latter is impossible. Thus, $C_\varphi(D) = 0$. The proof of the second condition is obvious.

Theorem 2. *If, under the assumptions of Theorem 1, we have: 1) for* $n \geq 2$ *the set* F *is the sum of a finite number of closed domains whose boundaries satisfy the Poincaré condition* ([1])*; 2) for* $n = 1$ *the set* F *is the sum of a finite number of closed and bounded domains, then the equilibrium measure* μ^* *for* $\varphi(t) \in \Phi_1$ *is such that: a)* $u^*(P) \geq W(F)$ *everywhere on* F*; b)* $u^*(P) = W(F)$ *everywhere on* F_{μ^*}*; c)* $u^*(P)$ *is continuous everywhere in* R_n*; d)* $u^*(P) \leq 2^h W(F)$ *everywhere in* R_n*.*

The proof of conditions a), b) follows from the fact that for φ-potentials generated by kernels $\varphi(t) \in \Phi_1$, Lemma 3 holds, and therefore one can apply the method of proof proposed by Frostman in ([1]). Conditions c), d) follow respectively from Lemma 2 and Lemma 1.

We shall say that a kernel $\varphi(t)$ satisfies the **weak Frostman maximum principle** if, from the fact that the continuous φ-potential $u(P)$, generated by a measure μ, does not exceed some constant K on the support of this measure, it follows that it does not exceed this constant everywhere in R_n.

Theorem 3. *If, under the assumptions of Theorem 2, the kernel* $\varphi(t) \in \Phi_1$ *satisfies the weak Frostman maximum principle, then there exists an equilibrium measure* μ^* *for the given set* F *such that the* φ*-potential* $u^*(P)$ *generated by it has the following properties: 1)* $u^*(P) = W(F)$ *everywhere on the set* F*; 2)* $u^*(P) \leq W(F)$ *everywhere in* R_n*.*

The proof of this theorem follows immediately from the validity of the weak Frostman maximum principle and Theorem 2.

Theorem 4. *If, under the assumptions of Theorem 1, the kernel* $\varphi(t) \in \Phi_1$ *satisfies the weak Frostman maximum principle, then there exists an equilibrium measure* μ^* *such that: 1)* $u^*(P) = W(F)$ *almost everywhere, in the sense of* φ*-capacity, on* F*; 2)* $u^*(P) \leq W(F)$ *everywhere in* R_n*; 3) if* $u^*(P_0) = W(F)$*, then* $u^*(P)$ *is continuous at the point* P_0*.*

The proof of conditions 1 and 2 of Theorem 4 is not difficult to obtain by using Frostman' s method from ([1]). Condition 3 follows from condition 2 and from the lower semicontinuity of the potential $u^*(P)$.

We note that the set of points of F at which $u^*(P) < W(F)$ is a set of type F_σ.

Theorem 5. *For the existence of an equilibrium* φ*-potential* $u^*(P)$ *on a closed set* F *from* Π*, it is necessary and sufficient that the kernel* $\varphi(t) \in \Phi_1$ *satisfy the weak Frostman maximum principle.*

The sufficiency of the condition follows from Theorem 4. The proof of necessity is not difficult to obtain by using the method of work ([1]). From Theorem 5 follows the result of work ([5]).

§ 5. On the uniqueness of the equilibrium measure. Without loss of generality one may assume that Π is an n-dimensional parallelepiped whose edges are such that $-\pi \leq x_i \leq \pi,\ i = 1, 2, \dots, n$.

Theorem 6. *If the kernel $\varphi(t)$ has positive Fourier coefficients, then, under the assumptions of Theorem 4, there exists a unique equilibrium measure μ^* for the given set F.*

Proof. Let μ_1^* and μ_2^* be two equilibrium measures. Put $\chi = \mu_1^* - \mu_2^*$, then $I(\chi) = 0$. Let $u_1^*(P)$ and $u_2^*(P)$ be φ-

equilibrium potentials generated respectively by the measures μ_1^* and μ_2^*. Denote by $c^{(i)}_{k_1,\dots,k_n}$ the Fourier coefficients of $u_i^*(P)$, $i = 1, 2$. We have $c^{(i)}_{k_1,\dots,k_n} = d_{k_1,\dots,k_n}\gamma^{(i)}_{k_1,\dots,k_n}$, where $d_{k_1,\dots,k_n}$ are the Fourier coefficients of the kernel $\varphi(t)$, and $\gamma^{(i)}_{k_1,\dots,k_n}$ are the Fourier–Stieltjes coefficients of $d\mu_i^*$. For $n \geq 2$, putting

$$\tau(t) = \begin{cases} (1-t^2)^\delta & \text{for } 0 \leq t \leq 1, \\ 0 & \text{for } t \geq 1, \end{cases}$$

we obtain: if $\delta > (n-1)/2$, then the spherical means $S^\tau_{R,i}(P)$ of the Fourier series of $u_i^*(P)$ (see (6)) are such that

$$\lim_{R\to\infty} S^\tau_{R,i}(P) = 2^l\Gamma(l+1)u_i^*(P) \tag{9}$$

at every point of discontinuity of the function $u_i^*(P)$, i.e. almost everywhere in the sense of φ-capacity on F (since the sum of two Borel sets of φ-capacity zero is a set of φ-capacity zero). From the boundedness of $u_i^*(P)$ and from (9)

$$I(\varkappa) = \lim_{R\to\infty} \sum_{\rho^2 \leq R^2} \tau(\rho/R) d_{k_1,\dots,k_n} \left(\gamma^{(1)}_{k_1,\dots,k_n} - \gamma^{(2)}_{k_1,\dots,k_n}\right)^2 .$$

Hence we find that $\gamma^{(1)}_{k_1,\dots,k_n} = \gamma^{(2)}_{k_1,\dots,k_n}$ for all possible combinations of integral values k_j, $j = 1, 2, \dots, n$. From the uniqueness of the solution of the trigonometric moment problem (see (7)) it follows that $\mu_1^* = \mu_2^*$. For $n = 1$ the proof of Theorem 6 evidently follows from consideration of Fejér means.

p. 6. On the Fourier coefficients of functions depending on the radius.

Lemma 4. Let $n \geq 2$ and let $f(P)$ be a function integrable in the Lebesgue sense on Π and depending only on $r = \sqrt{x_1^2 + \dots + x_n^2}$ for $P = (x_1, \dots, x_n)$. Then its Fourier coefficients $d_{k_1,\dots,k_n}$ are such that

$$d_{k_1,\dots,k_n} = d(\rho),$$

where $\rho = \sqrt{k_1^2 + \cdots + k_n^2}$ and

$$d(\rho) = \frac{\chi(n)}{\rho} \int_0^{\pi\sqrt{n}} r^{l+1} f(r) J_l(r\rho)\, dr; \tag{10}$$

$\chi(n)$ is a positive function depending only on n, $l = (n-2)/2$, and $J_l(z)$ is the Bessel function.

Theorem 7. Let $n \geq 3$ and let the function $t^{n-2}\varphi(t)$ be strictly decreasing; then, for $\varphi(t) \in \Phi$, the kernel $\varphi(t)$ has positive Fourier coefficients.

The proof of this theorem follows from Lemma 4 for $n \geq 4$, and for $n = 3$ from expression (10).

Theorem 8. If $n \geq 2$ and $\varphi(t) \in \Phi$ is such that $\{-\Phi_t'(t)\}$ is strictly decreasing, then $\varphi(t)$ has positive Fourier coefficients.

This theorem is proved analogously to Theorem 7.

If the kernel $\varphi(t) \in \Phi_1$, then, when the conditions of Theorems 4 and 7 (or 8) are fulfilled, there exists a unique equilibrium measure for the given set F.

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