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Abstract

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PERIODIC AND ALMOST PERIODIC LIMITING REGIMES OF CONTROL SYSTEMS WITH SEVERAL, GENERALLY SPEAKING, DISCONTINUOUS NONLINEARITIES

(Presented by Academician L. S. Pontryagin, January 24, 1966)

1°. Consider the differential equations of a control system with $\varkappa$ nonlinear blocks $\varphi_j = \varphi_j(\sigma_j)$:

$$dx/dt = Px + q\varphi(\sigma) + f(t), \qquad \sigma = r^* x. \tag{1}$$

Here the matrices $P,\ q,\ r,\ x,\ f(t),\ \varphi(\sigma) = \|\varphi_j(\sigma_j)\|$ are real and have, respectively, dimensions $\nu \times \nu,\ \nu \times \varkappa,\ \nu \times \varkappa,\ \nu \times 1,\ \nu \times 1,\ \varkappa \times 1$ (and, consequently, $\sigma = \|\sigma_j\|$ is a matrix of order $\varkappa \times 1$). We shall assume that the $\varphi_j(\sigma_j)$ are piecewise continuous functions having only discontinuities of the first kind, and that

$$0 \le \Delta\varphi_j/\Delta\sigma_j \le \mu_j, \qquad j = 1, 2, \dots, \varkappa, \tag{2}$$

where $\Delta\varphi_j = \varphi_j(\sigma_j + \Delta\sigma_j) - \varphi_j(\sigma_j)$, $-\infty < \sigma_j < +\infty$, $-\infty < \Delta\sigma_j < +\infty$, and $\sigma_j,\ \sigma_j + \Delta\sigma_j$ are points of continuity of the function $\varphi_j(\sigma_j)$. Without loss of generality we shall suppose that $\mu_j < \infty$ for $j = 1, \dots, \varkappa_1$; $\mu_j = \infty$ for $j = \varkappa_1 + 1, \dots, \varkappa_1 + \varkappa_2 = \varkappa^*$. Suppose that $f(t) \in L(t_1, t_2)$ for any $-\infty < t_1 < t_2 < +\infty$.

We shall call a solution of system (1) any absolutely continuous $\nu \times 1$ matrix-function $x(t)$ such that, for some $\varkappa \times 1$ matrix-function $\psi(t) = \|\psi_j(t)\|$, summable on every interval and called an augmented function of $\varphi[\sigma(t)]$, the following holds almost everywhere:

$$dx/dt = Px + q\psi(t) + f(t), \qquad \varphi_j[\sigma_j(t) - 0] \le \psi_j(t) \le \varphi_j[\sigma_j(t) + 0],$$

where $\sigma(t) = \|\sigma_j(t)\| = r^* x(t)$. It is easy to prove a local existence theorem, as well as continuous dependence of the solution on the initial conditions in the following sense: if $x(t, a_k)$ are solutions of system (1) on a finite interval $\Delta = [t_0, T]$ or $\Delta = [T, t_0]$ and $x(t_0, a_k) = a_k \to a$ as $k \to \infty$, then for some subsequence $x(t, a_{k_n}) \to x(t)$ as $k_n \to \infty$, $t \in \Delta$, and $x(t)$ is a solution of system (1) for $t \in \Delta$. **We shall say that on (t_1, t_2) a solution $x(t)$ is a sliding regime if $\sigma(t) = r^* x(t)$ is not a point of continuity of the matrix-function $\varphi(\sigma)$ for any $t \in (t_1, t_2)$. For systems (1) of a certain class with**

$$f(t) = \sum_k \mathrm{Re}(f_k e^{i\omega_k t})$$

sliding regimes (on $(-\infty, +\infty)$) are found simply (see (2) for the case $\varkappa = 1$). From Theorem 1 below it follows that in a number of cases the sliding regime $x^0(t)$ is limiting, i.e., for any solution $x(t)$ one has $|x(t) - x^0(t)| \to 0$ as $t \to \infty$*.

* It is possible that $\varkappa_1 = 0$ or $\varkappa_2 = 0$. The number of discontinuous nonlinearities is, obviously, no greater than $\varkappa_2$. The case in which, instead of (2), $\mu_j' \leq \Delta\varphi_j / \Delta\sigma_j \leq \mu_j''$ holds reduces to the case under consideration by replacing $\varphi_j = \varphi_j - \mu_j' \sigma_j$ or $\varphi_j = \mu_j'' \sigma_j - \varphi_j$.

** For $\varkappa_2 = 1$ the definition of a solution given here coincides with the definition of A. F. Filippov (1), but requires, in addition, the specification of the functions $\psi_j(t)$. For $\varkappa_2 > 1$, not every Filippov solution (1) is a solution in the sense defined here. Analogous definitions and assertions are valid also in the more general case of the system

$$dx/dt = F(x, \varphi, t),$$

where $\varphi = \varphi(\sigma)$, $\sigma = r^* x$, and F is a continuous function, $x, \varphi, F(x, \varphi, t) \in L(t_1, t_2)$, $-\infty < t_1 < t_2 < +\infty$.

*** Since a sliding regime is a nonclassical solution, this fact justifies the definition of solution given above.

Denote by $\chi(\lambda) = r^*(P - \lambda I)^{-1} q$ the transfer matrix-function of the linear part of the system from the inputs $\varphi_1, \dots, \varphi_\chi$ to the outputs $(-\sigma_1), \dots, (-\sigma_\chi)$, by μ_d^{-1} the diagonal matrix with diagonal elements $\mu_1^{-1}, \dots, \mu_\chi^{-1}$ (where $\mu_i^{-1} = 0$ if $\mu_i = \infty$), and by τ_d the diagonal $\chi \times \chi$ matrix with diagonal elements $\tau_1, \dots, \tau_\chi$.* Put

$$\pi(\omega) = \tau_d \mu_d^{-1} + \mathrm{Re}\,[\tau_d \chi(i\omega)].$$

Represent the matrices q, $\pi(\omega)$ in the form $q = \|q_1, q_2\|$, $\pi(\omega) = \|\pi_{jh}\|$ $(j, h = 1, 2)$, where q_j are matrices of order $\nu \times \chi_j$, and π_{jh} are matrices of order $\chi_j \times \chi_h$ $(j, h = 1, 2)$.

Theorem 1. Suppose that P is a Hurwitz matrix, that the rank of the matrix q_2 is χ_2, and that, for some $\tau_1 > 0, \dots, \tau_\chi > 0$, one has $\pi(\omega) > 0$ for $-\infty < \omega < \infty$,

$$\lim_{\omega \to \infty} \omega^2 (\pi_{22} - \pi_{21} \pi_{11}^{-1} \pi_{12}) > 0.$$

Then:

a) for (1) exponential convergence holds, i.e., for some numbers $\gamma > 0,\ \varepsilon > 0$, for any $t \geq t_0$ and any solutions $x_1(t), x_2(t)$,

$$|x_1(t) - x_2(t)| \leq \gamma \exp[-\varepsilon(t - t_0)]\, |x_1(t_0) - x_2(t_0)|; \tag{3}$$

b) if $f(t)$ is a matrix-function bounded on $(-\infty, +\infty)$, then there exists a unique solution $x^0(t)$ bounded on $(-\infty, +\infty)$ (a limiting regime); c) if $f(t + T) \equiv f(t)$, then $x^0(t + T) \equiv x^0(t)$; d) if $f(t)$ is an almost-periodic matrix-function, then $x^0(t)$ is also an almost-periodic matrix-function.***

Theorem 2. Let the rank of the $\nu \times \chi\nu$-matrix $\|q, Pq, \dots, P^{\nu-1}q\|$ be equal to ν; let the roots of the equation $\det(P - \lambda I_\nu) = 0$ lie in the half-plane $\operatorname{Re}\lambda \leq -\varepsilon < 0$, and let $\pi(-\varepsilon + i\omega) \geq 0$ for some $\tau_1 \geq 0, \dots, \tau_\chi \geq 0$ and all $\omega,\ -\infty < \omega < +\infty$. Then, for some $\gamma > 0$, for any $t \geq t_0$ and any solutions $x_1(t), x_2(t)$, (3) holds, and the assertions b), c), d) of Theorem 1 are also valid.

2°. The proof of Theorems 1 and 2 uses the following algebraic propositions, the most essential part of which, for $\chi > 1$, was proved by V. M. Popov ([5]).**** For $\chi = 1$ analogous propositions were established in ([6],[7]) (see also ([8],[9])). Let $X = X^*,\ A,\ C = C^*,\ a,\ b,\ \rho = \rho^* \geq 0$ be real matrices, respectively of orders $\nu \times \nu,\ \nu \times \nu,\ \nu \times \nu,\ \nu \times \chi,\ \nu \times \chi,\ \chi \times \chi$. Put

$$G = C - A^*X - XA, \qquad g = -Xa - b, \qquad Q(X) = \begin{Vmatrix} G & g \\ g^* & \rho \end{Vmatrix} \tag{4}$$

and consider the problem of determining conditions for the existence of a matrix $X = X^*$ satisfying the inequality $Q(X) \geq 0$.

Introduce the notation:

$$A_\omega = A - i\omega I_\nu, \qquad a_\omega = A_\omega^{-1} a, \qquad \pi_0(\omega) = \rho + 2\operatorname{Re} b^* a_\omega + a_\omega^* C a_\omega.$$

We shall call the $\chi \times \chi$-matrix $\pi_0(\omega)$ the characteristic of the matrix function $Q(X)$.

Theorem 3. a) For the existence of a solution $X = X^*$ of the inequality $Q(X) \geq 0$, it is necessary that $\pi_0(\omega) \geq 0$ for $-\infty < \omega < +\infty$; b) if the rank of the $\nu \times \chi\nu$-matrix $\Phi = \|a, Aa, \dots, A^{\nu-1}a\|$ is equal to ν, then the preceding condition is also sufficient; c) if $C \leq 0,\ Q(X) \geq 0$, then the null space of the matrix X is contained in the subspace orthogonal

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* Here and below the following notation is used. I_k is the identity $k \times k$-matrix. An asterisk denotes Hermitian conjugation, $\operatorname{Re} N = \frac{1}{2}(N + N^*)$. The notation $H > 0$ means that H is a positive definite matrix. $A + B\begin{Vmatrix} A & 0 \\ 0 & B \end{Vmatrix}$.

** For $\chi_2 = 0$ the condition on the matrix q_2 and the last inequality are absent. For $\chi_1 = 0$ the last inequality becomes the inequality

$$\lim_{\omega \to \infty} \omega^2 \pi(\omega) > 0.$$

*** For $\chi_2 = 0$ Theorem 1 was formulated by the author without proof in (^{3}). For $\chi = 1$ Theorem 1 was proved in (^{2}), and a closely related assertion in (^{4}). Note that from (4) right-hand uniqueness follows: if $x_1(t_0) = x_2(t_0)$, then $x_1(t) = x_2(t)$ for $t \geq t_0$. Left-hand uniqueness may fail.

**** Namely assertion b) of Theorem 3, which is given below in a formulation somewhat different from (^{5}).

by the columns of the matrix $\Psi = \|b, A^*b, \dots, A^{*\nu-1}b\|$; if, in particular, A is a Hurwitz matrix and the rank of Ψ is ν, then $X > 0$.

Assume that ρ has the form $\rho = \|\rho_{jh}\|$ $(j, h = 1, 2)$, where the order of ρ_{jh} is $\varkappa_j \times \varkappa_h$, $\varkappa_1 + \varkappa_2 = \varkappa$, $\rho_{12} = \rho_{21}^* = 0$, $\rho_{22} = 0$*. Represent the matrices $\pi_0(\omega)$, a, b, g in the form $\pi_0(\omega) = \|\pi_{jh}\|$ $(j, h = 1, 2)$, $a = \|a_1, a_2\|$, $b = \|b_1, b_2\|$, $g = \|g_1, g_2\|$, where the order of π_{jh} is $\varkappa_j \times \varkappa_h$, and the orders of a_j, b_j, g_j are $\nu \times \varkappa_j$.

Theorem 4. *Assume that A is a Hurwitz matrix and that the rank of a_2 is $\varkappa_2$. For the existence of a solution of the inequality $Q(X) \geq 0$, where $Q(X)$ has the (maximally possible) rank $\nu + \varkappa_1$, it is necessary and sufficient that the following be fulfilled: (I) $\pi_0(\omega) > 0$ for $-\infty < \omega < +\infty$, (II) $\pi_\infty = \lim_{\omega\to\infty} \omega^2(\pi_{22} - \pi_{21}\pi_{11}^{-1}\pi_{12}) > 0$**.

3°. Proof of Theorem 3. a) We have

$$a_\omega^* G a_\omega = 2 \operatorname{Re} a_\omega^*(g + b) + a_\omega^* C a_\omega.$$

Putting $z_\omega^* = \|a_\omega^*, -I_\varkappa\|$, we obtain the identity

$$z_\omega^* Q(X) z_\omega \equiv \pi_0(\omega).$$

Therefore $\pi_0(\omega) \geq 0$. b) By Lemma 1 (6) (5) there exists a matrix $X = M = M^*$ such that $Q(M)$ has the form

$$Q(M) = \|L, K\|^* \|L, K\| \geq 0.$$

c) Let $Xn = 0$. Then $n^*Gn = n^*Cn = 0$, since $G \geq 0$, $C \leq 0$. Therefore $Gn = 0$, $Cn = 0$, $XAn = 0$, i.e., the null space of the matrix X is invariant with respect to A. Let ξ be an arbitrary $\varkappa \times 1$ vector, $y^* = \|n^*, \xi^*\|$. From the inequality

$$y^* Q(X) y = 2 \operatorname{Re} n^* g \xi + \xi^* \rho \xi \geq 0$$

it follows that $n^*g = 0$, $n^*b = 0$. Replacing n by $An, A^2n, \dots$, we obtain that the vector n is orthogonal to the columns of the matrix Ψ.

4°. Proof of Theorem 4. Necessity. From the relations $Q(X) \geq 0$, $\rho_{22} = 0$, it follows that $g_2 = 0$, $Q(X) = Q_1 + \|0\|$, where the $(\nu + \varkappa_1) \times (\nu + \varkappa_1)$-matrix Q_1 is obtained from $Q = Q(X)$ (see (4)) by replacing g by g_1 and ρ by ρ_{11}. We have (see 3°, a)

$$\pi_0(\omega) \equiv z_\omega^* Q z_\omega = u^* Q_1 u,$$

where $u^* = \|a_\omega^*, \delta_1^*\|$, $\delta_1 = \| -I_{\varkappa_1}, 0\|$ (δ_1 is a matrix of order $\varkappa_1 \times \varkappa$). Since, by assumption, $Q_1 > 0$, it follows that $\pi_0(\omega) \geq 0$. Since the rank of $\|a^*, \delta_1^*\|$ is $\varkappa$ and

$$u^* = \|a^*, \delta_1^*\| S,$$

where $S = A_\omega^{*-1} + I_{\varkappa_1}$ is a nonsingular matrix, the rank of u^* is $\varkappa$ and $\pi_0(\omega) > 0$. Since

$$\pi_{11} = \rho_{11} + O(\omega^{-1}), \qquad \pi_{12} = g_1^* a_2 (i\omega)^{-1} + O(\omega^{-2}), \qquad \pi_{22} = a_2^* G a_2 \cdot \omega^{-2} + O(\omega^{-3}),$$

we have

$$\pi_\infty = a_2^* (G - g_1 \rho_{11}^{-1} g_1^*) a_2.$$

Since the rank of a_2 is $\varkappa_2$, $\rho_{11} > 0$, $Q_1 > 0$, it follows that $\pi_\infty > 0$.

Sufficiency. Suppose first that the rank of Φ is ν. It is enough to show that, in the preceding notation, the inequality

$$Q(X) \geq \varepsilon (I_\nu + \rho),$$

where the number $\varepsilon > 0$, has a solution $X = X^*$. The last inequality is equivalent to the inequality $Q(X) \geq 0$, in which C and ρ are replaced respectively by $C - \varepsilon I_\nu$ and $\rho - \varepsilon\rho$. According to item b) of Theorem 3, a solution exists if

$$\alpha(\omega) \equiv \pi_0(\omega) - \varepsilon\rho - \varepsilon a_\omega^* a_\omega \geq 0.$$

The validity of the last inequality for sufficiently small $\varepsilon > 0$ follows from the conditions $\rho_{11} > 0$, $\pi_0(\omega) > 0$, $\pi_\infty > 0$. Here one should use the fact that the inequality

$$\alpha = \|\alpha_{jh}\| > 0$$

(where α_{jh} are matrices of order $\varkappa_j \times \varkappa_h$, $j, h = 1, 2$) is equivalent to the inequalities

$$\alpha_{11} > 0, \qquad \alpha_{22} - \alpha_{21} \alpha_{11}^{-1} \alpha_{12} > 0.$$

Let the rank of Φ be $\nu_1 < \nu$. We reduce the problem to an analogous one in which, instead of A, a, b, there will be

$$A^S = S^{-1} A S = \left\| \begin{matrix} A_1 & B \\ 0 & A_2 \end{matrix} \right\|, \qquad a^S = S^{-1} a = \left\| \begin{matrix} a' \\ 0 \end{matrix} \right\|, \qquad b^S = S^* b = \left\| \begin{matrix} b' \\ b'' \end{matrix} \right\|, \tag{5}$$

and the rank of the matrix

$$\Phi_1 = \|a', \ldots, A_1^{\nu_1 - 1} a'\|$$

is ν_1. (Moreover, the number of rows of the matrices A_1, B, a', b' is equal to ν_1.) It is easy to construct, using Theorem 2 (^{10}), a nonsingular matrix S such that A^S, a^S have the form (5) and the rank of Φ_1 is ν_1. We replace the inequality $Q(X) \geq 0$ by the equivalent one

$$S_0^* Q(X) S_0 \geq 0,$$

where $S_0 = S + I_\varkappa$. From (4) it follows that

$$S_0^* Q(X) S_0 = Q^S(X^S),$$

where $X^S = S^*XS$ and $Q^S(X^S)$ has the form (4) with the replacement of A, C, a, b, X by $A^S, C^S = S^*CS, a^S, b^S, X^S$. From (5) it follows that $Q^S(X^S)$ and $Q(X)$ have identical characteristics: $\pi^S(\omega) \equiv \pi_0(\omega)$. Define the matrix X^S. Let

$$X^S = \left\| \begin{matrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{matrix} \right\|, \quad Q^S(X^S) = \left\| \begin{matrix} G_{11} & G_{12} & g^{(1)} \\ G_{21} & G_{22} & g^{(2)} \\ g^{(1)*} & g^{(2)*} & \rho \end{matrix} \right\|, \quad Q_{11}(X_{11}) = \left\| \begin{matrix} G_{11} & g^{(1)} \\ g^{(1)*} & \rho \end{matrix} \right\|,$$

where the matrices $X_{jh}, G_{jh}, g^{(j)}$ have, respectively, orders $\nu_j \times \nu_h$, $\nu_j \times \nu_h$, $\nu_j \times \chi$, $\nu_2 = \nu - \nu_1$. It is easy to verify that $Q^S(X^S)$ and $Q_{11}(X_{11})$ have identical characteristics. Since the rank of Φ_1 is ν_1, by what was proved above there exists $X_{11} = X_{11}^0$ such that $Q_{11}(X_{11}^0) \geq 0$ and the rank of $Q_{11}(X_{11}^0)$ is equal to $\nu_1 + \chi_1$. From the form of ρ it follows that the last (right-hand) χ_2 columns of $Q_{11}(X_{11}^0)$ are zero. Since an analogous condition must obviously also be satisfied for the matrix $Q^S(X^S)$, represent $g^{(2)}$ in the form $g^{(2)} = \|g_1^{(2)}, g_2^{(2)}\|$, where $g_j^{(2)}$ have orders $\nu_2 \times \chi_j$, and require that $g_2^{(2)} = 0$. We have $g^{(2)} = -X_{21}a_2' - b_2''$, where a_2', b_2'' are matrices composed of the last χ_2 columns of the matrices a', b''. The rank of a_2', coinciding with the rank of a_2, is equal to χ_2. Therefore we take $X_{21} = X_{21}^0 = -b_2''(a_2'^* a_2')^{-1} a_2'^*$. (If $\chi_2 = 0$, then X_{21}^0 is an arbitrary matrix.) In the matrix $Q^S(X^S)$ only G_{22} has remained undetermined. Let $[Q^S(X^S)]$, $[Q_{11}(X_{11})]$ be the matrices obtained from $Q^S(X^S), Q_{11}(X_{11})$ by deleting the χ_2 extreme right (zero) columns and the χ_2 lower (zero) rows. Since $[Q_{11}(X_{11})] > 0$, there exists $G_{22}^0 > 0$ such that, after replacing G_{22} by G_{22}^0, $[Q^S(X^S)] > 0$ is fulfilled. Since A_2 is a Hurwitz matrix and $G_{22} = F - A_2^* X_{22} - X_{22} A_2$, where F is a matrix depending only on the already found X_{11}^0, X_{12}^0, there exists a solution $X_{22} = X_{22}^0$ of the equation $G_{22} = G_{22}^0$. Let $X^S = \|X_{jh}^0\|$, $j, h = 1, 2$. Then $Q^S(X^S) \geq 0$ and the rank of $Q^S(X^S)$ is equal to $\nu + \chi_1$. This implies the assertion of the theorem for $\nu_1 < \nu$.

5°. **Proof of Theorem 1.** a) Estimate (3) holds if there exists a matrix $H = H^* > 0$ such that, for the function $V(y) = y^*Hy$, with $y = x_1(t) - x_2(t)$,

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* This assumption does not restrict generality. Indeed, let the rank of the matrix $\rho \geq 0$ be $\varkappa_1$. Replacing the inequality $Q \geq 0$ by the equivalent one $S^*QS \geq 0$, where $S = I_\nu + \delta$ and δ is a suitably chosen nonsingular $\varkappa \times \varkappa$ matrix, we obtain an analogous inequality in which ρ has the indicated form.

** For $\varkappa_1 = 0$ the matrix π_∞ has the form $\pi_\infty = \lim_{\omega \to \infty} \omega^2 \pi_0(\omega)$. For $\varkappa_2 = 0$ condition (II) is absent. For $\varkappa = 1$, $C = 0$, Theorem 4 was proved in ([8]). The case $C \neq 0$ reduces to the case $C = 0$ (see ([6])).

one has $\dot V \le -2\varepsilon V$. The expression $\dot V$ is transformed to the form $\dot V = -\Omega_1 - \Omega_2$, where

$$\Omega_1 = y^* G y + 2y^* g\psi + \psi^* \tau_d \mu_d^{-1} \psi, \quad G = -P^* H - HP, \quad g = -Hq - \tfrac{1}{2} r\tau_d,$$

$$\Omega_2 = (\sigma^{(1)} - \sigma^{(2)})^* \tau_d \psi - \psi^* \tau_d \mu_d^{-1} \psi, \quad \sigma^{(j)} = r^* x_j(t), \quad \psi = \psi^{(1)} - \psi^{(2)},$$

$\psi^{(j)}$ is the completed function $\varphi[\sigma^{(j)}]$. Using condition (2) and the inequalities $\varphi_n(\sigma_n^{(j)} - 0) \le \psi_n^{(j)} \le \varphi_n(\sigma_n^{(j)} + 0)$, $\tau_h > 0$, we obtain that $\Omega_2 \ge 0$. By Theorem 4, if the conditions of Theorem 1 are satisfied, there exists a matrix $H = H^*$ such that Ω_1 is a nonnegative form in y and ψ, having rank $\nu + \chi_1$. Therefore $G > 0$, $H > 0$, $\Omega_1 \ge 2\varepsilon V$ for sufficiently small $\varepsilon > 0$. Consequently, $\dot V(y) \le -2\varepsilon V(y)$. b) As above, we obtain that outside a sufficiently large ball $\dot V(x) < -\varepsilon V(x)$ holds for any solution $x = x(t)$. Assertion b) follows from Lemmas 1 and 2 (2), using the property indicated above of continuous dependence of the solution on the initial data. Assertions c), d) are proved in the same way as in paper (2).

6°. **Proof of Theorem 2** repeats the proof of Theorem 1. The only difference is that the existence of a matrix $H = H^*$, for which $\dot V(y) \le -2\varepsilon V(y)$ (with prescribed $\varepsilon > 0$), follows from Theorem 3.

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REFERENCES

1. A. F. Filippov, *Matem. sborn.*, **51** (93), no. 1 (1960).
2. V. A. Yakubovich, *Avtomatika i telemekh.*, **25**, no. 7 (1964).
3. V. A. Yakubovich, Proceedings of the International Conference on Multidimensional and Discrete Automatic Control Systems, Prague, 1965.
4. B. N. Naumov, Ya. Z. Tsypkin, *Avtomatika i telemekh.*, **25**, no. 6 (1964).
5. V. M. Popov, *Rev. Roumaine sci. techniques*, **9**, no. 4, 629 (1964).
6. V. A. Yakubovich, *DAN*, **143**, no. 2 (1962).

7. R. E. Kalman, *Proc. Nat. Acad. Sci. U.S.A.*, **49**, 201 (1963).
8. S. Lefschetz, *Stability of Nonlinear Control Systems*, N. Y.–London, 1965.
9. M. A. Aizerman, F. R. Gantmakher, *Absolute Stability of Regulated Systems*, Publishing House of the Academy of Sciences of the USSR, 1964.
10. F. R. Gantmakher, *Matrix Theory*, ch. 7, § 3, Moscow, 1954.

Note: Figure translations are in progress. See original paper for figures.

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