ON THE REGULAR REALIZATION IN THE LARGE OF TWO-DIMENSIONAL METRICS OF NEGATIVE CURVATURE

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Abstract

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ON THE REGULAR REALIZATION IN THE LARGE OF TWO-DIMENSIONAL METRICS OF NEGATIVE CURVATURE

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1. The article investigates the question of the regular realization in the large of an infinite strip π in a complete metric W^- of negative curvature K. The simplest such strip is formed in the following way. Let τ be some geodesic and let $\{g\}$ be a family of geodesics orthogonal to τ. On each geodesic of this family we lay off, on one side of τ, a segment of length a. The locus of the endpoints of these segments forms an equidistant curve $\bar{\tau}$. The part of W^- enclosed between τ and $\bar{\tau}$ is an infinite strip in the metric W^-. One can imagine strips of more complicated structure. For this one should carry out the construction indicated above, using instead of the geodesic τ some other line which is situated in W^- as a topological straight line. For example, the part of the Lobachevsky plane enclosed between two equidistant horocycles is an infinite strip.

The metric in each infinite strip π can be given by means of the line element

$$d\bar{s}^2 = dx^2 + B^2(x, y)\, dy^2. \tag{1}$$

Here the function $B(x, y)$ is given in the strip $\Pi_a = \{0 \leq x \leq a,\ -\infty < y < \infty\}$ of the plane of the parameters x and y.

We shall assume that the following are satisfied.

Regularity conditions. The function $B(x, y)$ is a $C^{4,1}$-bounded function in the strip Π_a*, and the curvature K lies between two negative constants in the strip Π_a.

Theorem 1. *Under the regularity conditions formulated above, the metric in the strip π is realizable in E^3 by a surface of class $C^{3,1}$.*

The following theorem follows from Theorem 1.

Theorem 2. *Let W^- be a complete two-dimensional metric of negative curvature belonging to the class $C^{4,1}$**. Then the metric in any geodesic circle of the metric W^- can be realized in E^3 by a surface of class $C^{3,1}$.*

Indeed, any such circle can be enclosed in a strip of the type indicated above.

Remark 1. If in the conditions of Theorems 1 and 2 the regularity requirements on the metric are raised, then this metric can be realized by means of a surface of correspondingly greater regularity.

Remark 2. The surfaces by means of which the metrics indicated in Theorems 1 and 2 are realized are strongly twisted. More

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* We shall say that a function $f(x, y)$ is $C^{n,1}$-bounded in the strip Π_a if $f(x, y) \in C^{n,1}$ in Π_a and, in addition, this function, its derivatives up to order n inclusive, and the Lipschitz constants of the derivatives of order n are bounded in Π_a. Similarly one introduces the notion of a $C^{n,1}$-bounded function on a line.

** We shall say that a complete metric W^- of negative curvature belongs to the class $C^{n,1}$ if it can be given by means of a line element of the form (1) and the function $B(x, y) \in C^{n,1}$ on the whole plane.

more precisely, inside a cylinder of any prescribed radius one can indicate a surface by means of which the metric under consideration is realized.

Remark 3. Let a complete metric W without conjugate points, belonging to a sufficiently high regularity class, be given on the plane (no conditions are imposed on the sign of the curvature of the metric W). Then the metric in any geodesic disk of the metric W can be asymptotically realized in the following sense. Let the metric W belong to the class C^{2n}. There exists a family of regularly realizable metrics W_ε, depending on the parameter ε and approximating the metric of the geodesic disk under consideration with accuracy up to $O(\varepsilon^{2n})$. At the same time, the surfaces S_ε by means of which the metrics W_ε are realized are regular strongly twisted surfaces (they are located inside cylinders whose radii are $C\varepsilon$, where C is some constant). We give an example of the simplest such asymptotic realization. Let the metric W of class C^n be defined by the line element (1).* Then the metrics W_ε of class C^{n-1}, defined by the line element

$$ds_\varepsilon^2 = (1 + \varepsilon^2 B_x^2)\, dx^2 + 2\varepsilon^2 B_{xB} y\, dx\, dy + (B^2 + \varepsilon^2 B_y^2)\, dy^2, \tag{2}$$

approximate the metric of any geodesic disk of the metric W in the class C^{n-1} with accuracy up to $O(\varepsilon^2)$. The line element (2) is the line element of the surface S_ε given by the parametric equations

$$X = \varepsilon B \cos y/\varepsilon, \qquad Y = \varepsilon B \sin y/\varepsilon, \qquad Z = x. \tag{3}$$

It is evident that the surface S_ε is located inside the cylinder with axis oZ of radius $C\varepsilon$, where $C > \max B$.

2. Let the line element $ds^2 = E\,dx^2 + 2F\,dx\,dy + G\,dy^2$ have negative Gaussian curvature $K = -k^2$. Instead of the coefficients l, m, n of the second form, normalized so that $ln - m^2 = -k^2$, we shall consider the Riemannian invariants $r(x, y)$, $s(x, y)$ (if the line element ds^2 is realized by means of a regular surface, then r and s are the angular coefficients of the images of the asymptotic lines in the parametric plane xoy). Since

$$l = 2krs/(s-r), \qquad m = -k(r+s)/(s-r), \qquad n = 2k/(s-r), \tag{4}$$

from the Peterson-Codazzi and Gauss equations we obtain the system**

$$s_x + rs_y = A_1 + A_2 r + A_3 s + A_4 s^2 + A_5 rs + A_6 rs^2,$$

$$r_x + sr_y = A_1 + A_3 r + A_2 s + A_4 r^2 + A_5 rs + A_6 r^2 s, \tag{5}$$

where the A_i, hereafter called the **coefficients of the right-hand sides of system (5)**, are expressed in terms of Γ^k_{ij} and derivatives of the function $Q = \frac{1}{2}\ln k$.

Remark 4. System (5), generally speaking, is not equivalent to the system of the Peterson-Codazzi and Gauss equations. However, if a solution $\{r(x, y), s(x, y)\}$ of system (5) satisfies the condition $r \neq s$, then l, m, n, defined by the relations (4), are solutions of those equations.

3. **Lemma 1.** *Let the coefficients of the right-hand sides of system (5) be $C^{1,1}$-bounded functions in the strip Π_a, and let the initial data $r_0(y)$ and $s_0(y)$ be $C^{1,1}$-bounded functions on the axis oy. Then in some strip $\Pi_h = \{0 \le x \le h,\ -\infty < y < \infty\}$ there exists a $C^{1,1}$-bounded solution $\{r(x, y), s(x, y)\}$ of system (5), reducing to $\{r_0(y), s_0(y)\}$ at $x = 0$. For the width h of this strip there exists an estimate*

* A complete metric W without conjugate points, given on the plane and belonging to a sufficiently high regularity class, can be defined by means of a line element of the form (1).

** For a line element of the form (1), such a system was obtained by B. L. Rozhdestvenskii [1].

from below, depending on the initial data and the coefficients of the right-hand sides of system (5).

A brief outline of the proof of an analogous assertion for more general systems, given in a finite domain, without an estimate of the size of the domain in which the solution exists, is given, for example, in Courant ([2]). For our purposes it is more convenient to consider systems of the form (5) given in a strip. In addition, an estimate from below of the width h of the strip Π_h is important.

Let the coefficients of the right-hand sides of the given system of the form (5) be $C^{1,1}$-bounded in the strip Π_a functions, and let $\{r, s\}$ be a $C^{1,1}$-bounded solution of this system in Π_a, determined by $C^{1,1}$-bounded initial data $\{r_0, s_0\}$ on *oy*. Under the assumptions made, the following lemmas are valid; they follow from Lemma 1 and certain additional considerations and express the correctness property of the given solution $\{r, s\}$.

Lemma 2. *One can indicate a $\delta > 0$ such that any system of the form (5) whose coefficients of the right-hand sides differ in Π_a, in the $C^{1,1}$ metric, from the coefficients of the right-hand sides of the given system by less than δ, has in this strip a $C^{1,1}$-bounded solution for any initial data that differ on the axis oy, in the $C^{1,1}$ metric, from $\{r_0, s_0\}$ by less than δ. The constant δ depends on the coefficients of the right-hand sides of the given system and on its given solution $\{r, s\}$.*

Let $\delta > 0$ be the constant determined in Lemma 2 for the given system of the form (5), its solution $\{r, s\}$, and initial data $\{r_0, s_0\}$. Consider two systems of the form (5) whose coefficients of the right-hand sides differ in Π_a, in the $C^{1,1}$ metric, from the coefficients of the right-hand sides of the given system by less than δ. For each of these two systems indicate initial data that differ in the $C^{1,1}$ metric from the initial data $\{r_0, s_0\}$ by less than δ.

Lemma 3. *If the coefficients of the right-hand sides of the two indicated systems differ in the strip Π_a, in the $C^{1,1}$ metric, by less than ε, and the initial data differ on the axis oy, in the $C^{1,1}$ metric, also by less than ε, then the solutions of these systems determined by the indicated initial data differ by less than $P\varepsilon$. The constant P depends on the coefficients of the right-hand sides of the given system and its solution $\{r, s\}$.*

4. We now indicate the outline of the proof of Theorem 1. For the line element (1), system (5) is homogeneous ($A_1 \equiv 0$). Therefore in any strip Π_a there is the solution $r \equiv 0,\ s \equiv 0$. Let $\delta > 0$ be the number corresponding to this solution by Lemma 2. Consider the line element (2) approximating the line element (1). For the given $\delta > 0$ one can indicate an $\varepsilon_0 > 0$ such that for every $\varepsilon < \varepsilon_0$ the coefficients of the right-hand sides of system (5) for the element (2) will differ in the $C^{1,1}$ metric from the coefficients of system (5) for the element (1) by less than δ. Since the line element (2) is realized by means of the surface (3), the solution $\{r_\varepsilon, s_\varepsilon\}$ of system (5), constructed for the element (2), can be found in explicit form, since r_ε and s_ε are the angular coefficients of the images of asymptotic lines in the parametric plane *xoy*. It is easy to verify that

$$r_\varepsilon = \varepsilon k + O(\varepsilon^2), \qquad s_\varepsilon = -\varepsilon k + O(\varepsilon^2), \qquad k = \sqrt{-K}. \tag{6}$$

Let ε be any positive number satisfying the condition $\varepsilon < \varepsilon_0$. Then, according to Lemma 2, system (5), constructed for the element (1), has in Π_a a solution $\{r, s\}$ determined by the initial data $r_0(y) = r(0, y) = r_\varepsilon(0, y)$, $s_0(y) = s(0, y) =$

$s_\varepsilon(0, y)$. The coefficients of the right-hand sides of the systems (5) for the elements (1) and (2) differ in the strip Π_a, in the $C^{1,1}$ metric, by less than $C\varepsilon^2$ ($C = \text{const}$). Therefore, according to Lemma 3, the solutions $\{r, s\}$ and $\{r_\varepsilon, s_\varepsilon\}$ differ by less than $P\varepsilon^2$ ($P = \text{const}$).

* We shall say that two functions differ in Π_a in the $C^{1,1}$ metric by less than δ, if these functions, the corresponding derivatives, and the exact Lipschitz constants of these derivatives differ in Π_a by less than δ.

Hence, from relation (6) it follows that

$$r = \varepsilon k + O(\varepsilon^2), \qquad s = -\varepsilon k + O(\varepsilon^2). \tag{7}$$

It follows from (7) that $r \neq s$ for all sufficiently small ε. Using Remark 4, it is easy to complete the proof of Theorem 1.

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Note: Figure translations are in progress. See original paper for figures.