IMBEDDING THEOREMS FOR A CLASS OF FUNCTION SPACES

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Abstract

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V. A. DIKAREV

IMBEDDING THEOREMS FOR A CLASS OF FUNCTION SPACES

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If a function F belongs to the space W_p^l, then, according to S. L. Sobolev' s imbedding theorem [(1)], its derivatives of order l' ($l' < l$) are summable to the power p', where $l' - n/p' = l - n/p$ (n is the dimension of the space). This theorem cannot be improved while remaining within the framework of the spaces L_p; however, it turns out that one can obtain more information about derivatives of order l' if the space $L_{p'}$ is replaced by another space with a norm invariant under rearrangements of functions. Below is the minimal space with this property.

1. Let E^n be n-dimensional real space. For a measurable (complex) function f on E_n, introduce $f^*(t)$ ($0 < t < \infty$), the decreasing, right-continuous equimeasurable rearrangement of $|f|$:

$$\operatorname{mes}\{t : f^*(t) > N\} = \operatorname{mes}\{x : |f| > N\} \quad \text{for every } N.$$

We also introduce the function $f^{**}(t)$:

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s)\, ds,$$

the functional $\|\cdot\|_{\lambda,p}$

$$\|f\|_{\lambda,p} = \left(\int_0^\infty t^{p\lambda - 1} f^{*p}(t)\, dt \right)^{1/p}, \qquad 0 < \lambda < 1; \quad 1 \le p < \infty,$$

$$\|f\|_{\lambda,\infty} = \sup_{0<t<\infty} t^\lambda f^*(t), \qquad 0 \le \lambda < 1, \tag{1}$$

and the functional $\langle f \rangle_{\lambda,p}$, which is defined as $\|f\|_{\lambda,p}$, but with $f^{**}(t)$ substituted for $f^*(t)$ on the right-hand side of (1).

The inequality holds

$$C_1(\lambda, p)\|f\|_{\lambda,p} \le \langle f\rangle_{\lambda,p} \le C_2(\lambda, p)\|f\|_{\lambda,p}, \qquad 0 < C_1 < C_2 < \infty.$$

Following [2–4], by $L(\lambda, p, E_n)$ we shall denote the set of functions r for which $\|f\|_{\lambda,p} < \infty$; $L(\lambda, p, E_n)$ is a Banach space with norm $\langle\cdot\rangle_{\lambda,p}$.

Next, introduce the space $L(1, 1, E_n)$, coinciding with $L_1(E_n)$, by putting $\langle f\rangle_{1,1} = \|f\|_{1,1} = \|f\|_{L_1}$. It is obvious that for $\lambda = 1/p$ the space $L(\lambda, p, E_n)$ coincides with $L_p(E_n)$.

We shall also consider the space*

$$L(\lambda_1, \lambda_2, p, E_n) = L(\lambda_1, p, E_n) \cap L(\lambda_2, p, E_n), \qquad \lambda_1 \le \lambda_2.$$

$L(\lambda_1, \lambda_2, p, E_n)$ is a Banach space with norm

$$\|f\|_{\lambda_1,\lambda_2,p} = \max\{\|f\|_{\lambda_1,p}; \|f\|_{\lambda_2,p}\}.$$

* The necessity of considering the spaces $L(\lambda_1, \lambda_2, p, E_n)$ is due to the fact that below imbedding theorems are studied on the whole space. For a bounded domain the spaces $L(\lambda, p, E_n)$ suffice.

Theorem 1. Let one of the two conditions be satisfied:

a) $\lambda_1' > \lambda_1,\ \lambda_2' < \lambda_2$;

b) $\lambda_1' \ge \lambda_1,\ \lambda_2' \le \lambda_2,\ p' \ge p$.

Then $L(\lambda_1, \lambda_2, p, E_n) \subseteq L(\lambda_1, \lambda_2, p', E_n)$.

There are no other embedding relations for the spaces $L(\lambda_1, \lambda_2, p, E_n)$.

2. Lemma 1 is a strengthening of an inequality of V. P. Il' in ([6,7]) (in the form given to it by P. I. Lizorkin).

Lemma 1. Let

$$Jf = \int_{E_n} \frac{f(t)e^{-a\rho}}{\rho^\alpha}\, dt, \qquad f \in L(\lambda_1, \lambda_2, p, E_n), \qquad a > 0.$$

ρ is the distance between the points x and t.

If the conditions $0 < \alpha < n, \quad 1 - \alpha/n < \lambda_1 \le \lambda_2 < \min\{1, \frac{n-\alpha}{n-m}\}$ are satisfied, then $Jf \in L(\mu_1, \lambda_2, p, E_m)$, where E_m is a subspace of E_n, and

$$\mu_1 = (n/m)(\lambda_1 - 1 + \alpha/n).$$

Put $\lambda_1 = \lambda_2 = 1/p$. Then $f \in L_p$, $\quad Jf \in L(\mu_1, \lambda_2, p, E_m) = L(\mu_1, p, E_m) \cap L(\lambda_2, p, E_m)$. From the conditions of the lemma it follows that $\mu_1 < \lambda_1$, $p < 1/\mu_1$. Applying Theorem 1 to $L(\mu_1, p, E_m)$ with $\lambda_1' = \mu_1, \lambda_2' = 1/\mu_1$, we have

$$L(\mu_1, p, E_m) \cap L(\lambda_2, p, E_m) \subset L_{1/\mu_1} \cap L_p \subseteq L_{p'},$$

where $p \leq p' \leq 1/\mu_1$. Thus, in this particular case one obtains a strengthening of the Il' in-Lizorkin inequality.

3. By $L^r(\lambda_1, \lambda_2, p, E_m)$ we shall denote the set of functions F representable in the form

$$F(x) = \mathcal{F}^{-1}\big[(\mathcal{F} f)(1 + |\lambda|^2)^{-r/2}\big],$$

where

$$\mathcal{F} f = \int_{E_n} f(t) e^{i(\lambda, t)}\, dt$$

is the Fourier transform operator, $f \in L(\lambda_1, \lambda_2, p, E_n)$, $r \geq 0$. By $\|F\|^{(r)}_{\lambda_1, \lambda_2, p}$ we shall denote the norm in $L^r(\lambda_1, \lambda_2, p, E_n)$ of the function F. Put

$$\|F\|^{(r)}_{\lambda_1, \lambda_2, p} = \|f\|_{\lambda_1, \lambda_2, p}.$$

As is known ([5,6]), the function F is representable in the form of a "Bessel potential"

$$F(x) = \int_{E_n} G_r(x - y) f(y)\, dy,$$

where $f(y) \in L(\lambda_1, \lambda_2, p, E_n)$,

$$G_r(x) = \frac{|x|^{(r-n)/2}}{2^{(n+r-2)/2} \pi^{n/2} \Gamma(r/2)} \cdot K_{(n-r)/2}(|x|),$$

and $K_\alpha(t)$ is the Macdonald function.

We note that for $\lambda_1 = \lambda_2 = 1/p$ the spaces $L^r(\lambda_1, \lambda_2, p, E_n)$ coincide with the spaces L_p^r ([5,6]), while for $\lambda_1 = \lambda_2 = 1/p$ and integer $r = l$ they coincide with the spaces W_p^l of S. L. Sobolev.

We shall consider $F \in L^r(\lambda_1, \lambda_2, p, E_n)$ on the subspace E_m, $m \leq n$. The question is whether the function that arises in this way can be characterized in terms of the spaces $L^{r'}(\mu_1, \lambda_2, p, E_m)$. The answer is given by the following

Theorem 2. Let $F(x) \in L^r(\lambda_1, \lambda_2, p, E_n)$, $0 < \lambda_i < 1$, $r' \geq 0$, $r' - m\lambda_i < r - n\lambda_i < r'$ $(i = 1, 2)$. Then the embedding

$$L^r(\lambda_1, \lambda_2, p, E_n) \subset L^{r'}(\mu_1, \lambda_2, p, E_m)$$

holds, where

$$r' - m\mu_1 = r - n\lambda_1.$$

This theorem is proved with the aid of Lemma 1 and the theorem on multipliers of S. G. Mikhlin ([8]).

Put $\lambda_1 = \lambda_2 = 1/p$. Then $F(x) \in L_p^r(E_n)$ and

$$L_p^r(E_n) \subset L^{r'}(\mu_1, \lambda_2, p, E_m) = L^{r'}(\mu_1, p, E_m) \cap L^{r'}(\lambda_2, p, E_m),$$

where

$$\mu_1 = \frac{n}{mp} - \frac{r - r'}{m}.$$

Taking into account that $p < 1/\mu_1$, and applying Theorem 1 to $L^{r'}(\mu_1, p, E_m)$, we have

$$L^{r'}(\mu_1, p, E_m) \cap L^{r'}(\lambda_2, p, E_m) \subset L^{r'}_{1/\mu_1} \cap L^{r'}_p \subseteq L^{r'}_{p'},$$

where $p \leqslant p' \leqslant 1/\mu_1$. Thus, in this case we obtain a strengthening of P. I. Lizorkin' s theorem[6], which for integral r and r' coincides with S. L. Sobolev' s imbedding theorem.

Let again E_m be a subspace in E_n $(m \leqslant n)$, and let $F(x)$ range over the space $L^r(\lambda_1, \lambda_2, p, E_n)$. If r' satisfies the conditions of Theorem 2, then the representation

$$F(x) = \int_{E'_m} G_{r'}(x - y)\, \varphi(y)\, dy, \qquad x \in E_m. \tag{2}$$

is possible.

According to Theorem 2, $\varphi \in L(\mu_1, \lambda_2, p, E_m)$.

Theorem 3. *The set of functions $\{f(x)\}$, defined and measurable on E_m and such that $|f(x)|$ is majorized by a rearrangement of the modulus of some function $\varphi(x)$ occurring in the representation (2), coincides with $L(\mu_1, \lambda_2, p, E_m)$, where μ_1 is defined as in Theorem 2.*

Theorem 3 shows that Theorem 2 cannot, in a certain sense, be strengthened.

In conclusion I express my gratitude to V. I. Matsaev for posing the problem and for discussing the results.

Kharkov Institute of Mining Machine Building,
Automation and Computer Technology

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Note: Figure translations are in progress. See original paper for figures.