ON SOLUTIONS OF GENERAL LINEAR SYSTEMS OF EQUATIONS WITH PARTIAL DERIVATIVES OVER GEVREY SPACES

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Abstract

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ON SOLUTIONS OF GENERAL LINEAR SYSTEMS OF EQUATIONS WITH PARTIAL DERIVATIVES OVER GEVREY SPACES

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Definition 1. The **Gevrey space** $G(\delta, 1)$ (respectively $G(\delta, C)$) is the set of functions $\varphi(x,t)$, infinitely differentiable with respect to x and analytic (respectively continuous) with respect to t together with all their derivatives with respect to x, such that

$$|D_x^n \varphi(x,t)| \leq M(n!)^\delta / \rho^n, \qquad n = 0, 1, 2, \dots; \tag{1}$$

$M = M(\varphi), \rho = \rho(\varphi)$, in some domain $\mathfrak{M} \ni (x,t)$.

Definition 2. We shall call the system of equations

$$\sum_{k,j} A_{ij}^k(x,t) D_x^{{q_{ij}}^k} D_t^{{r_{ij}}^k} u_j(x,t) + b_i(x,t) = 0 \qquad (i, j = 1, 2, \dots, n) \tag{2}$$

quasi-hypoelliptic if it is solvable with respect to the highest derivatives in any of the variables. (This property is possessed, in particular, by hypoelliptic systems and equations.)

Definition 3. A system solvable with respect to the highest derivatives in only one of the variables will be called a **Cauchy system** in this variable.

Thus, the system

$$D_t^{p_i} u_i(x,t) = \sum_{k,j} A_{ij}^k(x,t) D_x^{{q_{ij}}^k} D_t^{{r_{ij}}^k} u_j(x,t) + b_i(x,t) \tag{3}$$

$$(i, j = 1, 2, \dots, n;\ p_i > r_{ij}^k)$$

will be a Cauchy system in the variable t.

For system (3) with Cauchy initial data

$$D_t^m u_i(x,t)\big|_{t=0} = \varphi_{mi}(x) \qquad (m = 0, 1, \dots, p_i - 1;\ i = 1, 2, \dots, n) \tag{4}$$

the Cauchy-Kovalevskaya theorem is known, asserting that if all $A_{ij}^k(x,t)$, $b_i(x,t)$, and $\varphi_{mi}(x)$ are analytic in x and t in a neighborhood of the point $(x_0, 0)$ and, moreover, $p_i \geq q_{ij}^k + r_{ij}^k$ (“normal system”), then in some neighborhood of this point there exists a unique solution of problem (3)—(4) analytic in x and t.

The Cauchy-Kovalevskaya theorem can also be generalized to nonanalytic systems. In particular, it can be shown that, for the existence of a unique solution of problem (3)—(4) analytic in t, it is sufficient that all $A_{ij}^k(x,t)$, $b_i(x,t)$, $\varphi_{mi}(x)$ belong to the space $G(\delta, 1)$. The quantity δ is called in this case, following G. S. Salekhov and V. R. Fridlender (^{1}), a **sufficient weight** of system (3).

Definition 4. By the quasicharacteristic polynomial of system (2) we shall mean the polynomial

$$\chi(s,\lambda) = \det \left\| \sum_k s^{{q_{ij}}^k} \lambda^{{r_{ij}}^k} \right\|.$$

The roots λ_i of the equation $\chi(s,\lambda) = 0$ can be written in the form of Puiseux series (2) in decreasing powers of s:

$$\lambda_i = c_{i0} s^{\theta_{i0}} + c_{i1} s^{\theta_{i1}} + \dots \quad (i = 1, 2, \dots, N;\ \theta_{i0} > \theta_{i1} > \dots).$$

Theorem 1. In order that problem (3)—(4) have a unique solution in the space $G(\delta, 1)$ (or $G(\delta, C)$), it is sufficient that: 1) the quantity $\delta = 1/\max_i \theta_{i0}$ be $\geqslant 1$; 2) all $A_{ij}^k(x,t)$, $b_i(x,t)$, $\varphi_{mi}(x) \in G(\delta, 1)$ (or $G(\delta, C)$).

The idea of the proof of Theorem 1 is as follows. By introducing new unknown functions, problem (3)—(4) is reduced to the problem

$$D_{tU} = A(x, t, D_x)U + B_0(x,t), \tag{3'}$$

$$U|_{t=0} = \Phi(x), \tag{4'}$$

where U, B_0, and Φ are column vectors, and A is a square matrix; all elements of B_0, Φ, and A belong to $G(\delta, 1)$ (or $G(\delta, C)$).

After integration, problem $(3')$—$(4')$ takes the form

$$U = \int_0^t A(x, t, D_x)U\, dt + \Phi + \int_0^t B_0(x,t)\, dt \equiv HU + F. \tag{5}$$

The formal solution of (5) can be written as the series

$$U = \frac{1}{1-H} F \equiv F + HF + H^2 F + \dots + H^{nF} + \dots , \tag{6}$$

where

$$H^{nF} = \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} A(x, t_1, D_x) A(x, t_2, D_x) \dots A(x, t_n, D_x) F(x, t_n)\, dt_n \equiv$$

$$\equiv \int_0^t L_n(x, t, D_x) F(x, t)\, dt^n, \qquad 0 \leqslant |t_n| \leqslant |t_{n-1}| \leqslant \dots \leqslant |t| \leqslant T.$$

Proceeding from the fact that the degree of the polynomial $L_n(x, t, D_x)$ with respect to D_x satisfies the inequality $[L_n] \geqslant n\theta + c$, where $\theta = \max_i \theta_{i0}$ and c is a certain constant, it is shown that, under the assumptions of the theorem, in some domain $|t| \leqslant T' \leqslant T$ one will have $|H^{nF}| \leqslant Cq^n,\ q < 1$ (the notation $|H^{nF}| \leqslant Cq^n$ means that each element of the matrix H^{nF}, in absolute value, does not exceed Cq^n); i.e., the series (6) converges, and the obtained solution $U(x, t)$ belongs to $G(\delta, 1)$ (or $G(\delta, C)$).

In the case when the variable x is multidimensional, $x = (x_1, \dots, x_m)$, the index δ of the space $G(\delta, 1)$ (or $G(\delta, C)$) will also be a vector quantity: $\delta = (\delta_1, \dots, \delta_m)$. Inequality (1) from Definition 1 will in this case naturally be replaced by

$$|D_{x_1}^{n_1} \dots D_{x_m}^{n_m}| \leqslant M\, (n_1!)^{\delta_1} \dots (n_m!)^{\delta_m} / \rho^{n_1 + \dots + n_m} \tag{1'}$$

Finding δ does not differ from the one-dimensional case if we restrict ourselves to vectors δ in which $\delta_1 = \delta_2 = \dots = \delta_m$. To find this

of the common magnitude δ_i, in constructing the quasi-characteristic equation we replace $D_{x_1}^{n_1} \dots D_{x_m}^{n_m}$ by $s^{n_1 + \dots + n_m}$.

In the general case* the vector δ is defined as follows. Suppose the characteristic equation is written in the form

$$\lambda^N + \sum_{i=1}^{N} \sum_{k} a_{ik} s_1^{{\alpha_{i1}}^k} \dots s_m^{{\alpha_{im}}^k} \lambda^{N-i} = 0. \tag{7}$$

Define the vectors $\theta_i^k = (\theta_{i1}^k, \dots, \theta_{im}^k)$ by the formulas $\theta_{ij}^k = \alpha_{ij}^k / i$. Then any vector satisfying the inequalities

$$(\delta, \theta_i^k) \leq 1, \qquad \delta \geq (1, 1, \dots, 1) \tag{8}$$

(for all i, k occurring in (7)) may be taken as a sufficient weight δ.

The existence of a vector δ satisfying the inequalities (8) may be taken as the definition of systems of Kovalevskaya type.

Remark 1. For systems whose coefficients do not depend on the variables x, the restriction $\delta \geq 1$ is not required.

Remark 2. In the work (^{3}) A. Friedman found, for a sufficient weight of the system (3), the expression

$$\delta_F = \min_{i,j,k} \frac{p_i - r_{ij}^k}{q_{ij}^k}.$$

It can be shown that the inequality $\delta_F \leq \delta$ always holds. In particular, for the system (3′), in which

$$A = \left\| \begin{matrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ q_1(D_x) & q_2(D_x) & q_3(D_x) & \cdots & q_N(D_x) \end{matrix} \right\|,$$

by Friedman we have

$$\delta_F = \min_k \frac{1}{[q_k]} \qquad ([q_k] \text{ is the degree of the polynomial } q_k(D_x)).$$

On the other hand, such a system, obviously, reduces to a single equation, for which

$$\delta = \min \frac{N-K}{[q_k]}.$$

The conditions necessary for a system of the form (2) to have a solution analytic in one of the variables are formulated for Cauchy systems with respect to another variable,

$$D_x^{q_i} u_i(x,t) = \sum_{k,j} A_{ij}^k(x,t)\, D_x^{{q_{ij}}^k} D_t^{{r_{ij}}^k} u_j(x,t) + b_i(x,t) \tag{9}$$

$$(i,j = 1,2,\dots,n; \qquad q_i > q_{ij}^k).$$

In this case one already considers the Puiseux expansions of the roots of the quasi-characteristic equation $\chi(s,\lambda) = 0$:

$$s_i = \bar{c}_{i0}\lambda^{\bar{\theta}_{i0}} + \bar{c}_{i1}\lambda^{\bar{\theta}_{i1}} + \dots \qquad (i = 1,2,\dots,\bar{N};\ \bar{\theta}_{i0} > \bar{\theta}_{i1} > \dots).$$

Theorem 2. *Suppose all the coefficients of the system (9) belong to the space* $G(\bar{\delta}, 1)$*, where* $\bar{\delta} = \max\{1, \bar{\theta}\}$*,* $\bar{\theta} = \max_i \bar{\theta}_{i0}$*.*

Then from the analyticity in t *of a solution of the system, together with all its derivatives with respect to* x *that enter into the right-hand side of (9), it follows that this solution belongs to the space* $G(\bar{\delta}, 1)$*.*

* This result was communicated to me by V. R. Fridlender.

It follows from Theorem 2, in particular, that the Cauchy problem (2)–(4) for a quasi-elliptic system, even with coefficients analytic in all variables, has no solution analytic in t if the initial data $\varphi_{mi} \in G(\bar{\delta}, 1)$.

We shall call the quantity $\bar{\delta}$ the **necessary weight** of system (2). For the proof, system (9), by introducing new unknown functions, is brought to the form

$$D_{xU} = A(x, t, D_t)U + B(x, t), \tag{9'}$$

analogous to $(3')$. From the conditions of Theorem 2 it follows that system $(9')$ has a solution $U(x, t)$ analytic in t. Successively estimating $D_x^{mU}(x, t)$, after a number of transformations with majorants we obtain

$$U(x, t) \in G(\bar{\delta}, 1).$$

In some cases, in the formulation of Theorem 2 one may dispense with the requirement that the derivatives of the solution be analytic in t, since this fact follows from the analyticity of the solution itself. For example, for the equation

$$D_x^q u = A(x, t) D_t^p u + B(x, t)$$

from the analyticity of the solution u in t there follows the analyticity of $D_x^q u$, whence, by applying Kolmogorov' s theorem $(^4)$, we also obtain the analyticity of $D_x^i u$ $(1 < i < q)$.

Remark 3. For systems whose coefficients do not depend on the variables x, the necessary weight is determined by the equality $\bar{\delta} = \max_i \bar{\theta}_{i0}$.

Remark 4. For systems which are not Cauchy systems with respect to x, Theorem 2 does not apply.

Remark 5. For quasi-elliptic systems, in determining the necessary weight of the system one may also use the formula

$$\bar{\delta} = \max\left\{1, \frac{1}{\min_i \bar{\theta}_{i0}}\right\}.$$

Hence, in particular, it is seen that $\bar{\delta} \geq \delta$.

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Note: Figure translations are in progress. See original paper for figures.