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MATHEMATICS

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THE FRATTINI SUBGROUP OF LINEAR GROUPS AND FINITARY APPROXIMABILITY

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Let G be an arbitrary group, and let $\mathfrak{M} = (M_\alpha)$ be the set of all proper maximal subgroups of G. The subgroup

$$\Phi(G) = G \cap_\alpha M_\alpha$$

is called the Frattini subgroup of the group G (if the set $\mathfrak{M}$ is empty, then $\Phi(G) = G$). According to the classical Frattini theorem, for a finite group G the subgroup $\Phi(G)$ is nilpotent.

The question naturally arose of the possibility of extending this theorem to a broader class of groups, at least to finitely generated groups. The most general known result belongs to P. Hall ([1]): the Frattini subgroup of a finite extension of a finitely generated metanilpotent group is nilpotent. At the same time, in ([1]) an example is constructed of a soluble finitely generated group whose Frattini subgroup is not nilpotent. Although for arbitrary finitely generated groups the Frattini subgroup need not be nilpotent, in the case of linear groups over a field the question remained open. It was formulated explicitly by M. I. Kargapolov ([2]), problem 32).

In the present note a positive solution of this question is given for a wider class of linear groups than finitely generated ones*. The proof uses the simplest number-theoretic considerations and is based on approximating the groups under consideration by finite linear groups; for this purpose Mal' tsev' s approximation theorem ([3]) is somewhat generalized and refined; in particular, the projective homomorphisms of the group are naturally induced by homomorphisms of the corresponding commutative rings.

Let G be a group; let (G_α) be the set of its images under homomorphisms $\varphi_\alpha : G \to G_\alpha$; let e_α be the identity of G_α. If, for every $e \neq g \in G$, $\varphi_\alpha(g) \neq e_\alpha$

except for a finite number of indices α, then G will be called a limit of the groups G_α: $G = \lim_{\rightarrow}(G_\alpha, \varphi_\alpha)$. We shall use analogous terminology also in the case of rings.

By P we denote the ground field, assumed arbitrary; P_0 is the prime subfield. For arbitrary $a_1, a_2, \dots, a_k \in P$, by $\Omega[a_1, a_2, \dots, a_k]$ we denote the subring of P generated by $1, a_1, a_2, \dots, a_k$.

We shall need some elementary facts on approximation of commutative rings of finite type by fields.

If Z is the ring of integers and F is a finite field of characteristic q, then by $Z[X] = Z[x_1, x_2, \dots, x_m]$, $F[X] = F[x_1, \dots, x_m]$ we denote the rings of commutative polynomials over Z and F, respectively. Then

$$Z[X] = \lim_{\rightarrow}(\Sigma_i, \varphi_i), \quad i = 1, 2, \dots,$$

* It is known that in an arbitrary linear group the Frattini subgroup may fail to be nilpotent.

where Σ_i is a finite field of characteristic q_i, $q_i \neq q_j$ for $i \neq j$;

$$F[X] = \lim_{\longrightarrow}(\Delta_i, f_i),$$

where Δ_i are finite fields of characteristic q. In the second case the fields Δ_i arise as the result of the natural homomorphisms

$$f_i: \quad F[X] \to \Omega[a_1^{(i)}, a_2^{(i)}, \dots, a_m^{(i)}],$$

where $a_1^{(i)}, a_2^{(i)}, \dots, a_m^{(i)}$ are elements of some finite extension F_i of the field F. In the case of the ring of integers one considers the composite homomorphism

$$\varphi_i: \quad Z[X] \to Z_{q_i}[X] \to \Omega[a_1^{(i)}, a_2^{(i)}, \dots, a_m^{(i)}],$$

where Z_{q_i} is the residue field of Z modulo the prime q_i. It is easy to see that any subring $\Omega[a_1, a_2, \dots, a_k] \subset P$ possesses an analogous approximation, for it is an algebraic extension of finite type of some polynomial ring $Q[a_1, a_2, \dots, a_t]$ over Z or F. By d we denote the dimension of $\Omega[a_1, a_2, \dots, a_k]$ over the quotient field of the ring $Q[a_1, a_2, \dots, a_t]$.

Lemma 1. a) If P_0 is a finite field, P_0^d is its extension of degree d, then for any infinite algebraic extension $R \supset P_0^d$

$$\Omega[a_1, a_2, \dots, a_k] = \lim_{\longrightarrow}(\Omega_i, f_i),$$

where the finite fields $\Omega_i \subset R$. b) If P_0 is infinite; P_0^d is an extension of degree d; $p_1, p_2, \dots, p_r$ are primes, then

$$\Omega[a_1, a_2, \dots, a_k] = \varinjlim(\Omega_i, \varphi_i),$$

where the order of Ω_i is equal to $q_i^{dm_i}$, where $(m_i, p_j) = 1$; $i = 1, 2, \dots$.

The proof follows from the preceding remarks, the countability of $\Omega[a_1, a_2, \dots, a_k]$, and the fact that a nonzero polynomial over an infinite field does not vanish identically. In case b) one considers infinite fields which are the union of finite fields of the required type.

Let $\mathfrak{R}_1, \mathfrak{R}_2$ be commutative rings, and let $GL(n, \mathfrak{R}_i)$ be the full linear group of degree n over $\mathfrak{R}_i$, $i = 1, 2$. Then a homomorphism $\varphi : \mathfrak{R}_1 \to \mathfrak{R}_2$ induces the natural homomorphism

$$\Psi_\varphi : \quad GL(n, \mathfrak{R}_1) \to GL(n, \mathfrak{R}_2).$$

Theorem 1. Let Γ be a linear group of degree n over $\Omega[a_1, a_2, \dots, a_k]$. Then

$$\Gamma = \varinjlim(\Gamma_i, \Psi_{\varphi_i}),$$

where Γ_i are linear groups of degree n over finite fields Ω_i, which can be chosen as in Lemma 1.

Proof. Let $\varphi_i : \Omega[a_1, a_2, \dots, a_k] \to \Omega_i$ be the natural homomorphisms, where Ω_i satisfy the requirements of Lemma 1. Let Ψ_{φ_i} be the homomorphisms of Γ into $GL(n, \Omega_i)$ induced by φ_i. For each $e \neq g \in \Gamma$ there exists Ψ_{φ_i} such that $\Psi_{\varphi_i}(g) \neq e$. Since the group Γ is countable, one constructs inductively such a sequence (Ψ_{φ_i}) that $\Psi_{\varphi_i}(\Gamma) = \Gamma_i$ satisfy the condition

$$\Gamma = \varinjlim(\Gamma_i, \Psi_{\varphi_i}).$$

Corollary (3). A finitely generated linear group

$$T = \varinjlim(T_i, \Psi_i),$$

where T_i are finite linear groups of degree n.

Indeed, T is contained in a linear group of degree n over $\Omega[a_1, a_2, \dots, a_k]$, where $\Omega[a_1, a_2, \dots, a_k]$ is generated by the elements of the generators and their inverses of the group T.

We shall need two simple number-theoretic lemmas.

Lemma 2. Let $p_1, p_2, \dots, p_r, q$ be prime numbers. Then for the set S of numbers of the form

$$s_i = \prod_{j=1}^{r} (p_j - 1) q^i, \qquad i = 1, 2, \dots,$$

there exist positive integers $\alpha_1, \alpha_2, \dots, \alpha_r$ such that

$$\delta_i = \prod_{t=1}^{n} (q^{ds_i t} - 1) = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} \omega_i,$$

where $(p_j, \omega_i) = 1,\ i = 1, 2, \dots;\ j = 1, 2, \dots, r$.

Proof. Let

$$c = \prod_{j=1}^{r} (p_j - 1),$$

$$q^{ds_i k} - 1 = (q^{dck} - 1) \left(\sum_{t=1}^{q_i} q^{dck(q^i - t)} \right).$$

Since $p_j / (q^{p_j - 1} - 1)$, it follows that $p_j / (q^{ck} - 1)$. At the same time

$$p_j \nmid \sum_{t=1}^{q^i} q^{dck(q^i - t)}$$

in view of the fact that $p_j \nmid q$. If

$$q^{dck} - 1 = p_1^{\beta_{1k}} p_2^{\beta_{2k}} \cdots p_r^{\beta_{rk}} \nu_k, \qquad (\nu_k, p_j) = 1,$$

then

$$\alpha_j = \sum_{k=1}^{n} \beta_{jk}, \qquad j = 1, 2, \dots, r.$$

Lemma 3. For any prime numbers $p_1, p_2, \dots, p_r$ there exists an infinite set Θ of prime numbers and positive integers $\alpha_1, \alpha_2, \dots, \alpha_r$ such that for every m satisfying the condition $(m, p_j) = 1,\ j = 1, 2, \dots, r$,

$$\delta_\theta = \prod_{t=1}^{n} (\theta^{dmt} - 1) = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} \omega_\theta,$$

where $(\omega_\theta, p_j) = 1$, for any $\theta \in \Theta$.

Proof. It is easily checked that the assertion of the lemma is valid if, as Θ, one takes the set of prime numbers of the form

$$1 + p_1 p_2 \cdots p_r + s p_1^2 p_2^2 \cdots p_r^2, \qquad s = 1, 2, \dots,$$

which is infinite by Dirichlet' s theorem on primes in an arithmetic progression.

Let now $p_1, p_2, \dots, p_r \neq q$ be all distinct prime divisors of the number $n!$; q the characteristic of the ground field; $q_1, q_2, \dots$ an infinite sequence of prime numbers distinct from p_j, $j = 1, 2, \dots, r$; d some positive integer; Δ_i a finite field whose order, for $q > 0$, is equal to $q^{d(p_1-1)\cdots(p_r-1)q^i}$, and for $q = 0$ is equal to $q_i^{dm_i}$, $(m_i, p_j) = 1$, $i = 1, 2, \dots$.

Lemma 4. If a linear group of degree n

$$G = \varinjlim (G_i, f_i),$$

where G_i are nilpotent linear groups of degree n over the fields Δ_i, which satisfy the conditions indicated above, then G is nilpotent.

Proof. It suffices to show that the nilpotency classes of the groups G_i are bounded in the aggregate. The group $G_i = R_i \times U_i$, where R_i is a completely reducible subgroup and U_i is unipotent. Since the nilpotency class of U_i does not exceed $n-1$, the groups G_i may be assumed completely reducible. If Z_i is the center of G_i, then the order of the group G_i/Z_i has the form

$$p_1^{\beta_1^{(i)}} p_2^{\beta_2^{(i)}} \cdots p_r^{\beta_r^{(i)}}$$

(see (4), p. 69). On the other hand, for $q > 0$ the order

G_i/Z_i is a divisor of the number

$$\delta_i = \prod_{t=1}^{n} \left(q^{ds_i t} - 1\right),$$

where $s_i = (p_1 - 1) \dots (p_r - 1) q^i$, and for $q = 0$ it is a divisor of the number

$$\gamma_i = \prod_{t=1}^{n} \left(q_i^{dm_i t} - 1\right).$$

It follows from Lemmas 2 and 3 that there then exist such $\alpha_1, \alpha_2, \dots, \alpha_r$ that $\beta_k^{(i)} < \alpha_k$ for all $i = 1, 2, \dots;\ k = 1, 2, \dots, r$. The latter means that the orders of the groups G_i/Z_i are bounded in the aggregate, which entails the boundedness of the nilpotency classes.

The main result of this note is the following.

Theorem 2. *Let G be a linear group of degree n over $\Omega[a_1, a_2, \dots, a_k]$. Then the Frattini subgroup $\Phi(G)$ is nilpotent.*

Proof. By Theorem 1,

$$G = \varinjlim(G_i, \Psi_i),$$

where G_i are linear groups of degree n over the finite fields Δ_i from Lemma 4. Such a choice of the fields Δ_i is possible by Lemma 1. If M_i is a maximal subgroup of G_i, then $H^{(i)} = \Psi_i^{-1}(M_i)$ is maximal in G. We denote by $\widetilde{\Phi}(G)$ the intersection of all such maximal subgroups of the group G. Clearly, $\Phi(G) \supseteq \widetilde{\Phi}(G)$. By construction,

$$\widetilde{\Phi}(G) = \varinjlim(\Phi(G_i), \widetilde{\Psi}_i),$$

where $\widetilde{\Psi}_i$ are the restrictions of Ψ_i to $\Phi(G)$. Since $\Phi(G_i)$ are nilpotent, $\widetilde{\Phi}(G)$ is also nilpotent by Lemma 4. The theorem is proved.

Corollary. *The Frattini subgroup of a finitely generated linear group is nilpotent.*

Remark 1. As is clear from the proof of Theorem 2, what has actually been proved is the nilpotency of the subgroup $\widetilde{\Phi}(G) \supseteq \Phi(G)$, which is the intersection not of all maximal subgroups, but only of the maximal subgroups of finite index. In general, $\widetilde{\Phi}(G) \neq \Phi(G)$.

Remark 2. In fact, Theorem 2 is true for any finitely generated commutative rings $\Omega[a_1, a_2, \dots, a_k]$ without nontrivial nilpotent elements, since under this condition $\Omega[a_1, a_2, \dots, a_k]$ is approximated by fields.

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Note: Figure translations are in progress. See original paper for figures.