ON AN ANALYTIC PROBLEM ARISING IN STATISTICS

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ON AN ANALYTIC PROBLEM ARISING IN STATISTICS

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In certain questions of the theory of statistical tests and unbiased estimates the following problem arises (see (1)). Let Π be an open bounded polycylinder in C^n. By $\mathscr{H}(\Pi)$ we denote the ring, under multiplication, formed by functions analytic in Π, and by $\theta(\Pi)$ its subring formed by functions admitting an estimate of the form*

$$|\varphi(z)| \leq c\rho^{-q}(z, C\Pi). \tag{1}$$

On $\theta(\Pi)$ we introduce the strongest of the topologies for which the set of functions satisfying inequality (1) with arbitrary fixed c and q is bounded. Let a be a certain matrix of size $t \times s$ (the numbers t and s are arbitrary), formed by functions analytic in Π. Consider the mapping of modules over the ring $\theta(\Pi)$

$$a : \theta^s(\Pi) \to \theta^t(\Pi), \tag{2}$$

which consists in multiplication by this matrix ($\theta^k(\Pi)$ is the direct sum of k copies of the ring $\theta(\Pi)$). The problem is to establish that the mapping (2) is a topological homomorphism, and, if possible, to describe its image in a simpler way.

For the space $\mathscr{H}(\Pi)$ the solution of the analogous analytic problem is well known. The corresponding theorem, due to H. Cartan (4), is as follows: in order that a function $\varphi \in \mathscr{H}^t(\Pi)$ belong to the subspace $a\mathscr{H}^s(\Pi)$, it is necessary and sufficient that the following be fulfilled

Condition (K). For every point $z \in \Pi$ the Taylor series of the function φ at this point belongs to the image of the mapping

$$a(z) : \mathfrak{S}^s \to \mathfrak{S}^t, \tag{3}$$

where $\mathfrak{S}$ is the space of formal power series with coefficients from C, and the action of the mapping $a(z)$ consists in multiplication by the Taylor series of the matrix a at the point z.** (In fact, the theorem established by Cartan applies to a broader class of domains and matrices.) It turns out that an analogous result is also valid for the space $\theta(\Pi)$.

Theorem 1. *In order that a function* $\varphi \in \theta^t(\Pi)$ *belong to the image of the mapping (2), it is necessary and sufficient that it satisfy condition (K). The mapping (2) is a topological homomorphism.*

Hence, in particular, it follows that for any function $\varphi \in \theta^t(\Pi)$ satisfying condition (K) and inequality (1), one can find a function ψ, analytic in Π, satisfying inequality (1) with certain constants c' and q', depending only on c and q (and also on Π and a), such that $a\psi = \varphi$.

We outline the course of the proof of Theorem 1, not depending on Cartan' s theorem. By $\mathfrak{A}_\varepsilon$ denote the space of power series absolutely convergent for $|\xi| \leqslant \varepsilon$, normed by means of the norm $\|\varphi\|_\varepsilon = \sup\{|\varphi(\xi)|,\ |\xi| \leqslant \varepsilon\}$.

Lemma 1 (see (2)). *For any point* $z \in \Pi$ *the identity operator* E *in the space* $\mathfrak{S}^t$ *admits the decomposition*

$$E = D(z) + a(z)G(z), \tag{4}$$

where $D(z)$ *is a (linear) operator in* $\mathfrak{S}^t$, $G(z)$ *is an operator acting from* $\mathfrak{S}^t$ *into* $\mathfrak{S}^s$, *and the operator* $D(z)$ *vanishes on the image (3). For any* ε, $0 < \varepsilon \leqslant 1$, *the operator* $G(z)$ *acts from the space* $\mathfrak{A}^t_\varepsilon$ *into* $\mathfrak{A}^s_{r\varepsilon}$, *and*

$$\|G(z)\varphi\|_{r\varepsilon} \leqslant \frac{1}{r\varepsilon^q}\|\varphi\|_\varepsilon, \qquad q > 0, \tag{5}$$

where the function $r = r(z)$ *is defined as follows: there exists a decreasing sequence*

$$\overline{\Pi} = N_0 \supset N_1 \supset N_2 \dots \supset N_m \supset N_{m+1} = \emptyset$$

of analytic varieties such that

$$r(z) = c\rho^q(z, N_{\nu+1}),$$

if $z \in N_\nu \setminus N_{\nu+1}$, $\nu = 0, \dots, m$, *and we put* $\rho(z, N_{m+1}) = 1$. *The operator* $D(z)$ *has an analogous property.*

Fix some numbers $\chi \geqslant 0$ and $0 < \lambda \leqslant 1$. Denote by π_z the open circular polycylinder with center at the point $z \in \Pi$, which is the product of disks of

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* $C\Pi$ is the complement of the set Π.

** In Cartan' s formulation of the theorem, instead of the space $\mathfrak{S}$ there appears the space of convergent power series. The equivalence of the two formulations follows from the fact that the space of convergent series is a Zariski ring (see (^{6})), whose completion coincides with $\mathfrak{S}$.

radius $\lambda\rho^{\chi}(z, C\Pi)$; by $U_{\chi,\lambda}(\Pi)$ the covering of the polycylinder Π formed by the domains π_z; by ${}^k\theta_{\chi,\lambda}$ the space of cochains of order k on the covering $U_{\chi,\lambda}(\Pi)$, whose coefficients are holomorphic functions equal to $O(\rho^{-q}(z, C\Pi))$ as $z \to C\Pi$. In ${}^k\theta_{\chi,\lambda}$ we introduce the topology in the natural way. The coboundary operator defines continuous mappings $\partial_k : {}^k\theta_{\chi,\lambda} \to {}^{k+1}\theta_{\chi,\lambda}$. The kernel of the operator ∂_0, obviously, coincides with $\theta(\Pi)$.

Lemma 2. *For any* χ, λ *and* $k > 0$ *there exists a continuous operator which assigns to a cocycle* $\varphi \in {}^k\theta_{\chi,\lambda}$ *a cochain* $\psi \in {}^{k-1}\theta_{\chi,\lambda/2}$ *such that* $\partial_{k-1}\psi = \varphi$.

For the proof this assertion is reduced, in the usual way, to a Cauchy-Riemann system in the class of all infinitely differentiable functions satisfying inequalities of the form (1), which is solved by means of convolution with the fundamental solution.

Inductive proposition. *Let* π *be a circular polycylinder of radius* δ *with center at the point* $z \in \Pi$ $(\delta < \rho(z, C\Pi))$. *For any function* φ*, holomorphic in* Π*, satisfying condition (K), one can find a function* ψ*, holomorphic in the concentric polycylinder* $\frac{1}{2}\pi$ *of radius* $\delta/2$*, such that* $a\psi = \varphi$ *and*

$$\sup_{\frac{1}{2}\pi} |\psi| \leqslant \frac{c}{[\delta\rho(z, C\pi)]^q} \sup_{\pi} |\varphi| \tag{6}$$

with some c, q*, depending only on* a.

We shall prove this assertion by induction on the homological dimension of the matrix a. By the homological dimension of the matrix a we shall mean the least number $d = d(a)$ such that for any circular polycylinder $\pi' \subset \Pi$ there exists an exact sequence of the form

$$0 \to \mathfrak{S}^d \xrightarrow{a_d(z)} \mathfrak{S}^{d-1} \xrightarrow{a_{d-1}(z)} \cdots \to \mathfrak{S}^1 \xrightarrow{a_1(z)} \mathfrak{S}^s \xrightarrow{a(z)} \mathfrak{S}^t, \qquad z \in \pi', \tag{7}$$

where $a_1, \dots, a_d$ are some matrices holomorphic in $\overline{\Pi}$. From the results (3) it follows that $d(a) < n$. If $a = 0$, we put $d(a) = -1$.

Thus, suppose that the inductive proposition has been proved for all matrices a' with $d(a') < d(a)$ (in the case $d(a) = -1$ it is obvious). We shall prove it for the matrix a. At each point $z \in {}^2/_3\Pi$ we apply the expansion (4) to the function φ. Since the function φ satisfies condition (K) and is analytic in the ${}^1/_3\delta$-neighborhood of z, we have $D(z)\varphi = 0$ and $\varphi = a(z)G(z)\varphi$, where $G(z)\varphi$ is a function analytic in the ${}^1/_2\delta r$-neighborhood of z. In the covering ${}^2/_3\Pi$ formed by these neighborhoods, one can inscribe a covering $N_{0,\lambda}({}^2/_3\Pi)$ with a suitable λ. The functions $G(z)\varphi$, considered on this subcovering, form a zero-order cochain ψ', and $a\psi' = \varphi$. Suppose that $\partial_0\psi' = 0$. Then from estimate (5) it follows that the function ψ' satisfies inequality (6), and consequently the inductive proposition is proved.

Suppose that $\partial_0\psi' \neq 0$. From the equality $a\,\partial_0\psi' = \partial_0\varphi = 0$ it follows that all coefficients of the cochain $\partial_0\psi'$ satisfy condition (K) with respect to the matrix

a_1 appearing in the sequence (7), written for $\pi' = \pi$. Clearly, $d(a_1) < d(a)$. Applying the inductive proposition, we can write the cochain ψ' in the form $a_1\chi$, where χ is a cochain on the covering $U_{0,\lambda/5}({}^2/{}_3\Pi)$ satisfying the inequality

$$|\chi| \leq [\lambda\rho(z, C\pi)]^{-q}|\psi'|$$

($|\chi|$ is the maximum of the moduli of the coefficients of the cochain χ). Suppose that $\partial_1\chi = 0$. Then, using a certain result close to Lemma 2, we can write the function χ in the form $\partial_0\chi'$, where χ' is a cochain on the covering $U_{0,\lambda/10}({}^3/{}_5\Pi)$ satisfying the inequality

$$|\chi'| \leq [\lambda\rho(z, C\pi)]^{-q}|\chi|.$$

It is clear that the function $\psi = \psi' - a_1\chi'$ is the required one.

If $\partial_1\chi \neq 0$, then $a_2\partial_1\chi = 0$, and consequently the coefficients of the cochain $\partial_1\chi$ satisfy condition (K) with respect to the matrix a_2, etc. The process described terminates no later than the n-th step; therefore the required function ψ can always be constructed. Thus the inductive proposition is proved.

Let us prove Theorem 1. Let the function $\varphi \in \theta^t(\Pi)$ satisfy condition (K). According to Lemma 1, in the neighborhood of each point $z \in \Pi$ of radius $r(a)\rho(z, C\Pi)$ we have the equality $\varphi = a(z)G(z)\varphi$. In the covering Π formed by these neighborhoods, one can inscribe a covering $U_{\chi,\lambda}(\Pi)$ with suitable χ and λ. The functions $G(z)\varphi$ form a zero-order cochain ψ' on this covering, and, using inequality (5), it is not difficult to establish that this cochain belongs to ${}^0\theta^s_{\chi,\lambda}$. If $\partial_0\psi' = 0$, then the theorem is proved. If, however, $\partial_0\psi' \neq 0$, then, taking into account that $a\,\partial_0\psi' = 0$, and using the inductive proposition, we can write the cochain $\partial_0\psi'$ in the form $a_1\chi$, where $\chi \in {}^1\theta^s_{\chi,\lambda/5}$. Suppose that $\partial_1\chi = 0$. Then from Lemma 2 $\chi = \partial_0\chi'$, $\chi' \in {}^0\theta^{s_1}_{\chi,\lambda/10}$. Consequently, the difference $\psi = \psi' - a_1\chi'$ is the required function. Next we consider the case $\partial_1\chi \neq 0$ and continue the argument analogously to the proof of the inductive proposition. Theorem 1 is proved.

By $\mathscr{H}(\overline{\Pi})$ we denote the ring of functions analytic in $\overline{\Pi}$.

Corollary. The $\mathscr{H}(\overline{\Pi})$-module $\theta(\Pi)$ is flat.

For the proof it is enough to use the flatness criterion for modules from (5).

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REFERENCES

1. Yu. V. Linnik, DAN, 161, No. 3, 520 (1965).

2. V. P. Palamodov, Dissertation, Moscow State University, 1965.

3. A. Andreotti, H. Grauert, Bull. Soc. math. France, 90, No. 2, 193 (1962).

4. A. Cartan, Bull. Soc. math. France, 78, No. 1, 29 (1950).

5. A. Cartan, S. Eilenberg. *Homological Algebra*, IL, 1960.

6. O. Zariski, P. Samuel, *Commutative Algebra*, IL, 1963.

Note: Figure translations are in progress. See original paper for figures.