ON THE UNIQUENESS OF THE SOLUTION OF THE SECOND AND THIRD BOUNDARY-VALUE PROBLEMS FOR A SECOND-ORDER ELLIPTIC EQUATION

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ON THE UNIQUENESS OF THE SOLUTION OF THE SECOND AND THIRD BOUNDARY-VALUE PROBLEMS FOR A SECOND-ORDER ELLIPTIC EQUATION

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This note considers the question of uniqueness of the solution of the second and third boundary-value problems for a second-order elliptic equation in the class of bounded functions in the presence of boundary singular points, i.e., points for which neither the structure of the boundary nor the behavior of the solution as it approaches them is specified.

In the case of the first boundary-value problem this question has been completely studied: the first boundary-value problem has a unique solution in the class of bounded functions if the boundary conditions are specified everywhere outside a set of zero capacity. This condition is necessary ([1,2]).

For the second and third boundary-value problems this question has been little studied. It is known that for domains having isolated singular points on the boundary, the solution turns out to be unique.

We shall call a boundary set **inessential** for the given boundary-value problem if the boundary conditions prescribed outside it determine the bounded solution uniquely. The main result of the note is that, for $n > 2$, an inessential set for the second and third boundary-value problems is a closed set E of zero $(n-2)$-dimensional Hausdorff measure, and for $n = 2$, of zero logarithmic measure. One can give an example showing that these conditions are quite close to being necessary.

A closed set E is said to have zero k-dimensional Hausdorff measure if, for every $\varepsilon > 0$, there exists a covering of E by balls of radii $r_1, \dots, r_N$ such that

$$\sum_{i=1}^{N} r_i^k < \varepsilon.$$

A closed set E is said to have zero logarithmic measure if, for every $\varepsilon > 0$, there exists a covering of E by disks of radii $r_1, \dots, r_N$ such that

$$\sum_{i=1}^{N} \frac{1}{\ln(1/r_i)} < \varepsilon.$$

Let D be a bounded domain of n-dimensional space, $X = (x_1, \dots, x_n)$, $n > 2$ $(n = 2)$, with boundary Γ. Suppose that in D the equation

$$\sum_{i,k=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ik}(x) \frac{\partial u}{\partial x_k} \right) + c(x)u = 0, \qquad c(x) \le 0, \tag{1}$$

is defined, with respect to the coefficients of which it is assumed that: 1) all coefficients are bounded; 2) $a_{ik}(x)$ are continuously differentiable; 3)

$$\sum_{i,k=1}^{n} a_{ik}(x)\xi_i\xi_k \ge \alpha \sum \xi_i^2, \qquad \alpha > 0.$$

Theorem 1. Let $E \subset \Gamma$ be a closed set of zero $(n-2)$-dimensional Hausdorff measure (zero logarithmic measure). Outside E, let the boundary Γ of the domain D be smooth. Suppose that equation (1) is defined in D, and let u be a solution of this equation, bounded in D, differentiable up to the ⋯

$\Gamma \setminus E$ such that $\partial u/\partial \nu|_{\Gamma\setminus E} = 0$, where $\partial/\partial\nu$ is differentiation with respect to the conormal

$$\left(\frac{\partial}{\partial \nu} = \sum a_{ik}\gamma_i \frac{\partial}{\partial x_k}, \quad \gamma_i \text{ are the direction cosines of the normal} \right).$$

Then $u \equiv \text{const}$ in D.

Theorem 2. Let $E \subset \Gamma$ be a closed set of zero $(n-2)$-dimensional Hausdorff measure (zero logarithmic measure). Outside E, the boundary Γ of the domain D is smooth. Let u be a solution of equation (1), bounded in D, differentiable up to $\Gamma \setminus E$, and such that $\partial u/\partial\nu + \beta(x)u|_{\Gamma\setminus E} = 0$; $\beta(x) \ge 0$ is a bounded measurable function, different from zero on a set of positive measure belonging to $\Gamma \setminus E$.

Then $u \equiv 0$ in D.

Lemma. Let a domain G be situated in the spherical layer $\{r_1 < |x| < r_2\}$, $x = (x_1, \dots, x_n)$; $n \ge 2$. The intersections of the boundary of the domain G with the spheres $|x| = r_1$ and $|x| = r_2$ are both nonempty. Let a quadratic form $\sum a_{ik}(x)\xi_i\xi_k$ be defined in the domain G, satisfying the inequality

$$\sum_{i,k=1}^{n} a_{ik}(x)\xi_i\xi_k \geq \alpha \sum_{i=1}^{n} \xi_i^2, \quad \alpha > 0, \tag{2}$$

and let the coefficients of this quadratic form be continuously differentiable functions in G. Let, in the closed domain $\overline{G}$, there be given a twice continuously differentiable function $u(x)$ satisfying the condition $\operatorname{osc}_{x\in\overline{G}} u(x) < 1$.

Then there exists such a piecewise smooth surface Σ, separating in G the sphere $|x| = r_1$ from $|x| = r_2$, that

$$\int_{\Sigma} \left|\frac{\partial u}{\partial \nu}\right| d\sigma < r_2^{n-2} \frac{C}{\ln(r_2/r_1)},$$

where C is a constant depending on the constant α of inequality (2) and on n, the dimension of the space.

Proof. Map the spherical layer $r_1 \leq |x| \leq r_2$ onto the lateral surface of a cylinder in the $(n+1)$-dimensional space $Y = (y_1, \dots, y_n, y_{n+1})$ by means of the transformation

$$y_i = x_i/|x|, \quad i = 1, \dots, n; \qquad y_{n+1} = \ln|x|.$$

Denote by $\widetilde{G}$ the image of the domain G under this transformation. Put $\tilde{u}(y) = u(x)$. Applying to the domain $\widetilde{G}$ and the function $\tilde{u}$ the method used by M. L. Gerver and E. M. Landis in paper (3), one can prove that on the surface of the cylinder there exists a piecewise smooth surface $\widetilde{\Sigma}$, separating in $\widetilde{G}$ the upper base of the cylinder from the lower one, such that

$$\int_{\widetilde{\Sigma}} \left|\frac{\partial \tilde{u}}{\partial \tilde{\nu}}\right| d\tilde{\sigma} < \frac{C}{\Delta},$$

where $\Delta = \ln(r_2/r_1)$ is the height of the cylinder, into the lateral surface of which the spherical layer is transformed; C is a constant depending on the constant α of inequality (2), and on n, the dimension of the space; $\tilde{\nu}$ is the direction into which, under the transformation, the direction ν of the conormal passes, and $d\tilde{\sigma}$ is the element of $(n-1)$-dimensional area of the surface $\widetilde{\Sigma}$. Denoting by Σ the preimage of $\widetilde{\Sigma}$, and taking into account that $\partial/\partial\tilde{\nu} = r\,\partial/\partial\nu$ and $d\tilde{\sigma} = \frac{1}{r^{n-1}}\,d\sigma$, we obtain

$$\int_{\Sigma} \left|\frac{\partial u}{\partial \nu}\right| d\sigma < r_2^{n-2} \frac{C}{\ln(r_2/r_1)}.$$

Proof of the theorem for $n = 2$. Let $u \not\equiv \text{const}$ in D. Consider the set D_a of points $(x_1, x_2) \in D$ where $u(x_1, x_2) > a$. Draw the level set $u = a$, choosing

a so that $\partial u/\partial\nu|_{\Gamma_a} < 0$, where ν is the conormal, which can be done by virtue of the following theorem of Kronrod-Landis (4).

Let $f(x_1, x_2)$ be a twice differentiable function of two variables, defined in some plane domain D. Denote by E_t

the level set $f(x_1, x_2) = t$. Then, for almost all t in the interval $(\inf_D f, \sup_D f)$, the level set E_t contains no points at which the gradient of the function f is zero.

Choose an arbitrary $\varepsilon > 0$ and cover the set E by a system of disks with radii $r_{11}, \dots, r_{1N}$ such that

$$\sum_{i=1}^{N} \frac{1}{\ln(1/r_{1i})} < \frac{\varepsilon}{2C},$$

where C is the constant in the lemma. For each disk $|x| = r_{1i}$, construct the concentric circle of radius $r_{2i} = \sqrt{r_{1i}}$. Consider the resulting annuli $K_i = \{r_{1i} < |x| < r_{2i}\}$. To each of the annuli K_i one may apply the lemma and, in accordance with it, construct a line L_i dividing the annulus and such that

$$\int_{L_i} \left|\frac{\partial u}{\partial \nu}\right| dl < \frac{C}{\ln(r_{2i}/r_{1i})} = \frac{2C}{\ln(1/r_i)}.$$

Then

$$\int_{\bigcup_{i=1}^{N} L_i} \left|\frac{\partial u}{\partial \nu}\right| dl < 2C \sum_{i=1}^{n} \frac{1}{\ln(1/r_{1i})} < \varepsilon.$$

Put

$$L = \bigcup_{i=1}^{N} L_i.$$

Let $\varepsilon > 0$ be so small that in D_a there is a point A not belonging to any of the disks we have constructed of radius r_{2i}. Denote by D'_a the set of points belonging to D_a and not separated from A by the curve L. The boundary of D'_a consists of a part of the boundary Γ—denote this part by Γ', a part of Γ_a—denote this part by Γ'_a, and a part of L, which we denote by L'.

Consider the equality

$$\int_{D_a} \left[\sum_{i,k=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ik}(x)\frac{\partial u}{\partial x_k}\right) + c(x)u\right] dx = 0.$$

Apply Green' s formula to the left-hand side of this equality. We obtain

$$\int_{\Gamma'} \frac{\partial u}{\partial \nu}\,dl + \int_{\Gamma'_a} \frac{\partial u}{\partial \nu}\,dl + \int_{L'} \frac{\partial u}{\partial \nu}\,dl + \int_{D'_a} \frac{\partial u}{\partial \nu}\,dx = 0. \tag{3}$$

By the choice of Γ_a, $\partial u/\partial\nu|_{\Gamma_a} < 0$; therefore

$$\int_{\Gamma'_a} \frac{\partial u}{\partial \nu}\,dl < 0,$$

and as ε decreases the absolute value of this integral increases.

$$\int_{\Gamma'} \frac{\partial u}{\partial \nu}\,dl = 0$$

by the hypothesis of the theorem.

$$\left|\int_{L'} \frac{\partial u}{\partial \nu}\,dl\right| < \varepsilon$$

by the construction of the curve L'.

$$\int_{D'_a} c(x)u\,dx \leq 0,$$

since $c(x) \leq 0,\ a > 0$. Taking

$$\varepsilon < \left|\int_{\Gamma'_a} \frac{\partial u}{\partial \nu}\,dl\right|,$$

we arrive at the conclusion that the left-hand side in equality (3) is a negative quantity. The contradiction obtained proves the theorem.

Proof of the theorem for $n > 2$. Let $u \not\equiv \text{const}$ in D. Consider the set D_a of points $(x_1, \dots, x_n) \in D$ for which $u(x) > a$. Draw the surface $u = a$, choosing a so that $\partial u/\partial\nu$ exists and is negative almost everywhere on Γ_a, which can be done by virtue of the following theorem, proved by Dubovitskii ([5]).

Let $f(x_1, \dots, x_n)$ be an $(n-\nu)$-times differentiable function of n variables. Then the set of singular points of the level set $f(x_1, \dots, x_n) = t$ has zero $(n-\nu-1)$-dimensional measure for almost

all t. A point at which the gradient of the function $f(x_1, \dots, x_n)$ vanishes is called a special point.

Choose an arbitrary $\varepsilon > 0$ and cover the set E by a system of balls of radii $r_{1i}, \dots, r_{1N}$ so that

$$\sum_{i=1}^{N} r_{1i}^{n-2} < \varepsilon \frac{\ln 2}{C \cdot 2^{n-2}},$$

where C is the constant in the lemma.

For each ball $|x| = r_{1i}$ construct a ball concentric with it of radius $r_{2i} = 2r_{1i}$. Consider the spherical layers obtained, $\{r_{1i} < |x| < r_{2i}\}$. To each of the spherical layers the lemma can be applied and, in accordance with it, a surface Σ_i can be constructed which divides the spherical layer and is such that

$$\int_{\Sigma_i} \left|\frac{\partial u}{\partial \nu}\right| d\sigma < r_{2i}^{n-2} \frac{C}{\ln(r_{2i}/r_{1i})}.$$

Then

$$\int_{\bigcup_{i=1}^{N} \Sigma_i} \left|\frac{\partial u}{\partial \nu}\right| d\sigma < C \sum_{i=1}^{N} \frac{r_{2i}^{n-2}}{\ln(r_{2i}/r_{1i})} = C \sum_{i=1}^{N} \frac{2^{n-2} r_{1i}^{n-2}}{\ln 2} < \varepsilon;$$

put $\Sigma = \bigcup_{i=1}^{N} \Sigma_i$. Let ε be so small that in D_a there is a point A not belonging to any of the balls of radius r_{2i} constructed by us. Denote by D'_a the set of points belonging to D_a and not separated from A by the surface Σ. The boundary of D'_a consists of a part of the boundary Γ, denote this part by Γ'; of a part of Γ_a, denote this part by Γ'_a; and of a part of Σ, which we denote by Σ'.

Consider the equality

$$\int_{D'_a} \left[\sum \frac{\partial}{\partial x_i}\left(a_{ik}(x)\frac{\partial u}{\partial x_k}\right) + c(x)u\right] dx = 0.$$

Apply Green’ s formula to the left-hand side of this equality. We obtain

$$\int_{\Gamma'_a} \frac{\partial u}{\partial \nu} d\sigma + \int_{\Gamma'} \frac{\partial u}{\partial \nu} d\sigma + \int_{\Sigma'} \frac{\partial u}{\partial \nu} d\sigma + \int_{D'_a} c(x)u\, dx = 0. \tag{4}$$

By the choice of Γ_a, $\partial u/\partial \nu \leqslant 0$ on Γ_a, and moreover $\partial u/\partial \nu = 0$ only on a set of measure zero. Therefore

$$\int_{\Gamma'_a} \frac{\partial u}{\partial \nu} d\sigma < 0$$

and, as ε decreases, the absolute value of this integral increases.

$$\int_{\Gamma'} \frac{\partial u}{\partial \nu} d\sigma = 0$$

by the condition of the theorem.

$$\left|\int_{\Sigma'} \frac{\partial u}{\partial \nu} d\sigma\right| < \varepsilon$$

by the construction of the surface Σ',

$$\int_{D'_a} c(x)u\,dx \leqslant 0,$$

since $c(x) \leqslant 0$. Taking

$$\varepsilon < \left| \int_{\Gamma'_a} \frac{\partial u}{\partial \nu}\,d\sigma \right|,$$

we arrive at the conclusion that on the left in equality (4) there stands a negative quantity. The contradiction obtained proves the theorem.

Theorem 2 is proved similarly.

In conclusion I express my deep gratitude to E. M. Landis for guidance and help in this work.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.