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INVESTIGATION OF A CERTAIN GENERALIZED SYSTEM OF SINGULAR INTEGRAL EQUATIONS IN POGORZELSKI' S CLASS OF FUNCTIONS $\mathfrak{H}$

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V. Pogorzelski (^{1}, ^{2}) introduced into consideration a class of functions $\mathfrak{H}$ and proved a theorem on its transformation into itself by an integral of Cauchy type. These results were a generalization to the complex case of the corresponding results of A. I. Guseinov (^{3}). In the present paper one generalization of the indicated theorem of V. Pogorzelski is made, and the theorem proved is applied to the investigation of a generalized system of singular integral equations with shift.

Let a curve

$$L = \sum \widehat{c_\sigma c_{\sigma'}},$$

be given, consisting of a system of a finite number of directed closed and open arcs $\widehat{c_\sigma c_{\sigma'}}$, having no common points except, possibly, common endpoints (for a closed arc the endpoints c_σ and $c_{\sigma'}$ coincide). Suppose that the arcs $\widehat{c_\sigma c_{\sigma'}}$ have continuous tangents at every interior point, and one-sided tangents at the endpoints. Number in an arbitrary way the endpoints of the arcs $c_1, c_2, \dots, c_p$, which may be ordinary endpoints, corner points, or nodal points; the cusp case (return points) is excluded.

We give the definition of Pogorzelski' s class of functions $\mathfrak{H}_\alpha^\mu$ (^{1}, ^{2}) with respect to the system of arcs specified above.

Definition. By the class $\mathfrak{H}_\alpha^\mu$ we shall mean the set of all functions of a complex variable $\varphi(t)$, defined and continuous at every point $t \in L$ except, possibly, the points of discontinuity $c_1, c_2, \dots, c_p$, which satisfy the inequalities

$$|\varphi(t)| \prod_{\sigma=1}^{p} |t - c_\sigma|^\alpha \le \rho, \qquad W(t, t_1)|\varphi(t_1) - \varphi(t)| \le \chi |t_1 - t|^\mu, \tag{1}$$

where ρ and χ are arbitrary positive constants, and the real parameters α and μ satisfy the conditions $0 \le \alpha < 1$, $0 < \mu < 1$, $\alpha + \mu < 1$; the second of inequalities

(1) holds for each pair of interior points t and t_1 of an arbitrary arc $\widehat{c_\sigma c_{\sigma'}}$, and the auxiliary function $W(t, t_1)$ is defined as follows:

$$W(t, t_1) = \begin{cases} |t - c_\sigma|^{\alpha+\mu}, & \text{when } t, t_1 \in \widehat{c_\sigma c_{\sigma'}} \text{ and } c_\sigma = c_{\sigma'}, \\ |t - c_\sigma|^{\alpha+\mu} |t_1 - c_{\sigma'}|^{\alpha+\mu}, & \text{when } t, t_1 \in \widehat{c_\sigma c_{\sigma'}} \text{ and } c_\sigma \neq c_{\sigma'}. \end{cases} \tag{2}$$

If the points t and t_1 lie on an open arc $\widehat{c_\sigma c_{\sigma'}}$, it is assumed that $t_1 \in \widehat{t c_{\sigma'}}$; but if t and t_1 lie on a closed arc $\widehat{c_\sigma c_{\sigma'}}$, it is assumed that $|t - c_\sigma| \leq |t - c_{\sigma'}|$ and, moreover, that the ratio of the length of the arc $\widehat{t t_1}$ on which the common endpoint lies to the length of the entire arc does not exceed $1/2$.

By $\mathfrak{H}_\alpha^\mu(\rho, \chi)$ we denote the subset of those functions of the class $\mathfrak{H}_\alpha^\mu$ which satisfy condition (1) with prescribed values of ρ and χ. Suppose, further, that the complex function $s(t)$, defined for $t \in L$, satisfies the following conditions, which we shall call conditions (S): it maps each arc $\widehat{c_\sigma c_{\sigma'}}$ of the curve L onto itself one-to-one, preserving orientation; moreover,

$$0 < m_s \leq \left| \frac{s(t_1) - s(t)}{t_1 - t} \right| \leq M_s, \tag{3}$$

for every pair of points $t, t_1 \in L$; $m_s \leq 1$, $M_s \geq 1$ are positive constants. Then the following holds.

Theorem. *If the complex function $f(t, \tau)$, defined for*

$$t, \tau \in L_0 = L - \sum_{\sigma=1}^{p} c_\sigma,$$

is a function of the class $\mathfrak{H}_\alpha^\mu(\rho, \chi)$ $(\alpha > 0)$ with respect to the variable τ, and also satisfies the Hölder condition with respect to the variable t, i.e.

$$|f(t, \tau)| \prod_{\sigma=1}^{p} |\tau - c_\sigma|^\alpha \leq \rho, \tag{4}$$

$$|f(t, \tau) - f(t_1, \tau_1)| \, W(\tau, \tau_1) \leq \chi(|\tau - \tau_1|^\mu + |t - t_1|^{\mu_1}), \tag{5}$$

where $0 < \mu < \mu_1 \leq 1$, $\alpha + \mu < 1$, and the complex function $s(t)$, defined for $t \in L$, satisfies conditions (S), then the function $F(t)$, defined on the set L_0 by the integral

$$F(t) = \int_L \frac{f(t, \tau)}{\tau - s(t)} \, d\tau,$$

belongs to the class $\mathfrak{H}_\alpha^\mu(k_1\rho + k_2\chi,\ k_3\rho + k_4\chi)$, where k_1, k_2, k_3, k_4 are positive constants independent of the function $f(t, \tau)$.

Proof. Let $s(t) = t'$; then $t = s^{-1}(t')$ ($s^{-1}(t')$ is the inverse function of $s(t)$) and $f(t,\tau) = f[s^{-1}(t),\tau] = f^*(t',\tau)$; consequently,

$$F(t) = F^*(t') = \int_L \frac{f^*(t,\tau)}{\tau - t'}\, d\tau.$$

In view of (4) and (5) we have

$$|f^*(t,\tau)| \prod_{\sigma=1}^{p} |\tau - c_\sigma|^\alpha \le \rho, \tag{6}$$

$$|f^*(t',\tau) - f^*(t_1',\tau_1)|\, W(\tau,\tau_1) \le \frac{\chi}{m_s}(|t' - t_1'|^{\mu_1} + |\tau - \tau_1|^\mu). \tag{7}$$

Then, by V. Pogorzelski' s theorem (2), we have:

$$|F^*(t')| \prod_{\sigma=1}^{p} |t' - c_\sigma|^\alpha \le c_1\rho + c_2\frac{\chi}{m_s}, \tag{8}$$

$$|F^*(t') - F^*(t_1')|\, W(t',t_1') \le \left(c_3\rho + c_4\frac{\chi}{m_s}\right) |t' - t_1'|^\mu, \tag{9}$$

where c_1, c_2, c_3, c_4 are positive constants independent of the function $f(t,\tau)$.

By inequalities (2) and (3) we obtain

$$\frac{W(t',t_1')}{W(t,t_1)} \ge m_s^{2(\alpha+\mu)}, \qquad \frac{|t' - t_1'|}{|t - t_1|} \le M_s, \qquad \frac{\prod_{\sigma=1}^{p} |t' - c_\sigma|^\alpha}{\prod_{\sigma=1}^{p} |t - c_\sigma|^\alpha} \ge m_s^{p\alpha}. \tag{10}$$

Finally, taking into account (8), (9), and (10), we obtain

$$|F(t)| \prod_{\sigma=1}^{p} |t - c_\sigma|^\alpha \le k_1\rho + k_2\varkappa,$$

$$|F(t) - F(t_1)| W(t,t_1) \le (k_3\rho + k_4\varkappa)|t - t_1|^\mu,$$

where the constants independent of the function $f(t,\tau)$ are

$$k_1 = c_1 m_s^{-p\alpha}, \qquad k_2 = c_2 m_s^{-(p\alpha+1)}, \qquad k_3 = c_3 m_s^{\mu} m_s^{-2(\alpha+\mu)}, \qquad k_4 = c_4 m_s^{\mu} m_s^{-(2\alpha+2\mu+1)}.$$

The theorem is proved.

Next we study the nonlinear system of singular integral equations:

$$\varphi_k(t) = F_k\left[t, \varphi_1(t), \varphi_2(t), \dots, \varphi_n(t), \int_L \frac{N_{k1}\left[t, \tau, \varphi_1(\tau), \varphi_2(\tau), \dots, \varphi_n(\tau)\right]}{\tau - s_{k1}(t)}\, d\tau, \dots\right.$$

$$\left.\dots, \int_L \frac{N_{km}\left[t, \tau, \varphi_1(\tau), \varphi_2(\tau), \dots, \varphi_n(\tau)\right]}{\tau - s_{km}(t)}\, d\tau\right] \equiv \widehat{A}_k[\varphi_1(t), \varphi_2(t), \dots, \varphi_n(t)], \quad (11)$$

where $\varphi_1(t), \varphi_2(t), \dots, \varphi_n(t)$ are unknown functions, and L is the system of arcs defined above.

We shall study the system under the following assumptions.

I. The complex functions $F_k(t, u_1, u_2, \dots, u_n; u_{n+1}, \dots, u_{n+m})$, $k = 1, 2, \dots, n$, are defined in the domain $\Omega_1\{t \in L_0;\ u_j \in \Pi,\ j = 1, 2, \dots, n+m\}$, where Π is the open complex plane, and satisfy the inequality

$$|F_k(t, u_1, u_2, \dots, u_{n+m})| \le \frac{M_F}{\prod_{\sigma=1}^{p} |t - c_\sigma|^\alpha} + k_F \sum_{i=1}^{n+m} |u_i|, \quad (12)$$

as well as the generalized Hölder-Lipschitz condition

$$|F_k(t, u_1, \dots, u_{n+m}) - F_k(t_1, u_1', \dots, u_{n+m}')| \le \frac{k_F' |t - t_1|^\mu}{W(t, t_1)} + k_F \sum_{i=1}^{n+m} |u_i - u_i'|, \quad (13)$$

where M_F, k_F', and k_F are positive constants; $W(t, t_1)$ is the function defined by formula (2); the constants α and μ satisfy the conditions $\alpha > 0,\ 0 < \mu < 1,\ \alpha + \mu < 1$.

II. The complex functions $N_{k_i}(t, \tau, w_1, \dots, w_n)$, $k = 1, 2, \dots, n;\ i = 1, 2, \dots, m$, are defined in the domain $\Omega_2\{t, \tau \in L_0,\ w_j \in \Pi,\ j = 1, 2, \dots, n\}$, and satisfy the inequality

$$|N_{k_i}(t, \tau, w_1, \dots, w_n)| \le \frac{M_N}{\prod_{\sigma=1}^{p} |\tau - c_\sigma|^\alpha} + k_N \sum_{i=1}^{n} |w_i|$$

and the generalized Hölder-Lipschitz condition

$$|N_{k_i}(t,\tau,w_1,\dots,w_n)-N_{k_i}(t_1,\tau_1,w'_1,\dots,w'_n)| \leq \frac{k'_N\,[|\tau-\tau_1|^{\mu}+|t-t_1|^{\mu_1}]}{W(\tau,\tau_1)}+k_N\sum_{i=1}^{n}|w_i-w'_i|,$$

where M_N, k'_N, and k_N are positive constants; $W(\tau,\tau_1)$ is defined by formula (2), $\mu<\mu_1<1$.

III. The complex functions $s_{ki}(t)$, $k=1,2,\dots,n;\ i=1,2,\dots,m$, defined on L, satisfy the conditions (S) stated above.

The position of the points $t,t_1\in L_0$ and $\tau,\tau_1\in L_0$ is determined in accordance with the qualifications adopted in the definition of the class of functions $\mathfrak{H}_{\alpha}^{\mu}$.

To prove the existence of a solution of the system (11), we apply Schauder' s fixed-point principle. Consider the functional space Λ, consisting of all systems of n complex functions $[\varphi_1(t),\varphi_2(t),\dots,\varphi_n(t)]$, continuous on L_0 and satisfying the condition

$$\max_{1\leq k\leq n}\sup_{t\in L_0}\prod_{\sigma=1}^{p}|t-c_\sigma|^{\alpha+\mu}|\varphi_k(t)|<\infty.$$

In the usual way we define the sum of elements of Λ and the product of an element by a number:

$$[\varphi_1(t),\dots,\varphi_n(t)]+[\psi_1(t),\dots,\psi_n(t)]=[\varphi_1(t)+\psi_1(t),\dots,\varphi_n(t)+\psi_n(t)],$$

$$\lambda[\varphi_1(t),\dots,\varphi_n(t)]=[\lambda\varphi_1(t),\dots,\lambda\varphi_n(t)].$$

The norm $\|U\|$ of an element $U=[\varphi_1(t),\dots,\varphi_n(t)]$ of the space Λ is defined as follows:

$$\|U\|=\max_{1\leq k\leq n}\sup_{t\in L_0}\prod_{\sigma=1}^{p}|t-c_\sigma|^{\alpha+\mu}|\varphi_k(t)|.$$

Then the space Λ will be a Banach space. Next consider, in the space Λ, the set $Z(\rho,\chi)$ of all systems of n functions belonging to the set $\mathfrak{H}_{\alpha}^{\mu}(\rho,\chi)$. The set $Z(\rho,\chi)$ is evidently closed, convex, and compact (4).

Taking into account the form of the system (11), we transform the set $Z(\rho,\chi)$ by means of the operator

$$\psi_k(t) = \hat{A}_k[\varphi_1(t), \dots, \varphi_n(t)], \qquad k = 1, 2, \dots, n, \tag{14}$$

which assigns to each element $[\varphi_1(t), \varphi_2(t), \dots, \varphi_n(t)]$ from $Z(\rho, \chi)$ the element $[\psi_1(t), \dots, \psi_n(t)]$ of some set Z'.

Lemma 1. *If the constant K_F is sufficiently small, namely*

$$K_F < \min\left[\frac{1}{n(1 + mk_1k_N + mk_3K_N)}, \frac{1}{n(1 + mk_2K_N + mk_4K_N)}\right], \tag{15}$$

then $\rho = \rho_0$ and $\chi = \chi_0$ can be chosen so that the set Z' is a subset of the set $Z(\rho_0, \chi_0)$.

Lemma 2. *The operator (14) is continuous in the space Λ.*

As a result the following has been proved.

Theorem. *If conditions I—III are fulfilled and, in addition, inequality (15) holds, then the system of singular integral equations (11) has at least one solution in the class $\mathfrak{H}_\alpha^\mu$.*

Imposing on the derivatives of the real and imaginary parts of the function F_i conditions of the type (12) and (13), according to the scheme of the work (^{5}), one can prove the existence of a unique solution of equation (11), which is found by the method of successive approximations.

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CITED LITERATURE

[1] W. Pogorzelski, J. Math. and Mech. Indiana Univ., **9** (1960).
[2] W. Pogorzelski, *Rownania calkowe i ich zastosowania*, **3**, Warszawa, 1960.
[3] A. I. Guseinov, Izv. AN SSSR, ser. matem., **12**, 193 (1948).
[4] W. Zakowski, Zeszyty nauk. Politechn. Warsz., Matematyka, **2** (1964).
[5] W. Żakowski, Roczn. Polsk. towarz. mat., Prace Mat., **8** (1963).

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