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Abstract

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MATHEMATICS

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SOME PROBLEMS OF CONFORMAL MAPPING GENERATING QUADRATIC DIFFERENTIALS WITH FIVE DISTINCT POLES

(Presented by Academician M. A. Lavrent'ev, December 2, 1965)

Let $R \in (0, 1)$, $K_R : R < |z| < 1$, $C_r : |z| = r$ ($R \leq r \leq 1$). Denote by $\mathfrak{F}(R)$ the class of regular univalent mappings $f(z)$ of the annulus K_R , for which the bounded component of the complement of the domain $f(K_R)$ contains the points $w = 0$, $w = 1$, and the continuum $f(C_1)$. For $f \in \mathfrak{F}(R)$ let $\rho_R(f)$ denote the distance between $f(C_R)$ and $f(C_1)$. Denote by D_R the doubly connected domain conformally equivalent to the annulus K_R and having boundary consisting of two intervals of the real axis: $[-\infty, -t]$ and $[0, 1]$. The value t is a function of R , $t = t(R)$, strictly decreasing as R increases. Let $g_R(z)$ denote the function in $\mathfrak{F}(R)$ for which $g_R(K_R) = D_R$ and $g_R(1) = 1$. The following results have been established.

Theorem 1. In the class $\mathfrak{F}(R)$ the estimate $\rho_R(f) \geq t(R)$ is valid, and equality holds if and only if $f(z) \equiv g_R(ze^{i\theta})$ or $f(z) \equiv 1 - g_R(ze^{i\theta})$ (θ is a real number).

Let $\rho_{r_1, r_2}(f)$ denote the distance between $f(C_{r_1})$ and $f(C_{r_2})$.

Theorem 2. Let $R \leq r_1 < r_2 \leq 1$. Then in the class $\mathfrak{F}(R)$ the estimate

$$\rho_{r_1, r_2}(f) \geq -g_R(-r_1) + g_R(-r_2)$$

is valid, and equality holds if and only if $f(z) \equiv g_R(ze^{i\theta})$ or $f(z) \equiv 1 - g_R(ze^{i\theta})$ (θ is a real number). Moreover, for points $w_1 \in g_R(C_{r_1})$ and $w_2 \in g_R(C_{r_2})$ the equality

$$|w_1 - w_2| = -g_R(-r_1) + g_R(-r_2)$$

is fulfilled only when $w_1 = g_R(-r_1)$ and $w_2 = g_R(-r_2)$.

Suppose that on the z -sphere a triply connected domain B is given, whose boundary components are denoted by Γ_1, Γ_2 , and Γ , with Γ_1 not degenerating into a point. Let $\mathfrak{F}(B; \Gamma_1, \Gamma_2)$ be the class of univalent regular mappings $f(z)$ of the domain B , for each of which the complement $\Delta f(B)$ of the domain $f(B)$ satisfies the following conditions: the unbounded component of connectivity of the set $\Delta f(B)$ contains the set $f(\Gamma_2)$, and one of the bounded components of

connectivity of the set $\Delta f(B)$ contains the continuum $f(\Gamma_1)$ and the points $w = 0$ and $w = 1$. Denote by $\rho_f(B; \Gamma, \Gamma_1)$ the distance between the continua $f(\Gamma)$ and $f(\Gamma_1)$. In the class $\mathfrak{F}(B; \Gamma_1, \Gamma_2)$ there exists a (unique) mapping $g(z) \equiv g(B; \Gamma_1, \Gamma_2; z)$ onto a domain obtained from the w -plane by making three cuts, one of which coincides with the segment $[0, 1]$ of the real axis, and the other two are situated on the negative part of the real axis.

Theorem 3. In the class $\mathfrak{F}(B; \Gamma_1, \Gamma_2)$ the estimate

$$\rho_f(B; \Gamma, \Gamma_1) \geq \rho_g(B; \Gamma, \Gamma_1)$$

is valid, and the equality sign holds if and only if $f(z) \equiv g(z)$ or $f(z) \equiv 1 - g(z)$.

If one requires that some of the boundary components Γ and Γ_2 be degenerate, then the assertion of Theorem 3 can be reformulated for a doubly connected or simply connected domain (an annulus or a disk).

As consequences of the results obtained, one obtains solutions of the following problems.

1. Among univalent meromorphic mappings $f(z)$ of the annulus K_R (or of the unit disk), for which the preimage of the point $w = \infty$ serves

a fixed point $z \neq 0$, and the continuum $f(C_1)$ contains the points $w = 0$ and $w = 1$, find the mapping for which, for a given $r \in [R, 1)$, the distance $\rho_{r,1}(f)$ is minimal.

2. Find the maximum of the diameter of the boundary component $f(C_R)$ in the class of univalent conformal mappings $f(z)$ of the annulus K_R for which the unbounded component of the complement of the domain $f(K_R)$ contains the points $w = 0$, $w = \infty$, and the continuum $f(C_1)$, while the bounded component of this complement contains the point $w = 1$.
3. Among all doubly connected domains of the numerical plane whose boundary components have diameters not less than a given number $\lambda > 0$ and are at a distance from one another not exceeding unity, find the domain with the greatest Riemannian modulus (if the domain is conformally equivalent to the annulus K_R , then its Riemannian modulus is the number $1/R$).

In all the indicated problems the extremal images and domains have a boundary lying on one ray.

We note that the results obtained can be generalized to multiply connected (including infinitely connected) domains with one, two, or three distinguished boundary components, and also to quasiconformal mappings of such domains.

The principal feature of the results of the present communication is that the use of O. Teichmüller's principle (see ⁽¹⁾) leads here to quadratic differentials with five distinct poles. In this connection we observe that problems with a number of poles greater than three admit solution in exceptional cases, which, according to the way in which the main difficulty is overcome, may be divided

into several types: problems in which the poles can be brought onto one straight line by means of fractional-linear variation (see, for example, ^(2,3)); problems in which the poles can be brought onto one straight line by means of the method of symmetrization (see ⁽⁴⁻⁸⁾); problems whose solution has been obtained by the parametric method (see ^(9,10)). In almost all the indicated problems the number of poles is not greater than four. The results of the present communication in this respect constitute a new type of problem. Their solution is obtained by reducing to a problem (generating a quadratic differential with four distinct poles, one of which is double) on the minimum of the quantity $|f'(z_0)|$ in the class $\mathfrak{F}(R)$, for which, alongside the boundary variation of Schiffer ⁽¹¹⁾, it is possible to specify an additional special variation bringing the poles onto one straight line. We note that the possibility of the mentioned special variation is a consequence of the existence of a mapping extremal **simultaneously** in two different problems for the class Σ : in Löwner's problem ⁽¹²⁾ on the minimum of the quantity $|F'(z_0)|$ and in Faber's problem ⁽¹³⁾ on the maximum of the diameter of the boundary of the image. This point deserves mention also for the following reason. There exist unsolved problems analogous to the problems of the present work both in their formulation and in the supposed form of solution, but differing from them in that in the corresponding analogues of Löwner's and Faber's theorems there is no common extremal mapping. Therefore our method does not carry over to the problems mentioned, and at present we know of no other methods of solving the problems of the present work.

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