SOME PROPERTIES OF A SINGULAR INTEGRAL WITH DISCONTINUOUS DENSITY AND ITS APPLICATIONS

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Abstract

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MATHEMATICS

A. A. BABAEV

SOME PROPERTIES OF A SINGULAR INTEGRAL WITH DISCONTINUOUS DENSITY AND ITS APPLICATIONS

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1. Let Γ be a closed Jordan rectifiable curve, at each point of which a tangent exists, and let $S(t_1, t_2)$ denote the smaller of the lengths of the arcs joining the points $t_1, t_2 \in \Gamma$. Suppose that

$$S(t_1, t_2) \leqslant \beta(|t_1 - t_2|), \tag{1}$$

where $\beta(\delta)$ is a continuous, increasing function on $(0, l_0]$ (l_0 is the diameter of Γ); $\lim_{\delta\to 0} \beta(\delta) = 0$, and $\beta(\delta)/\delta$ is almost decreasing. Let $\alpha(\delta)$ be the inverse function of $\beta(\delta)$; l the length of the curve Γ. Denote by Φ the class of functions $\varphi(\delta)$, defined on $(0, l_0]$ and having the following properties: 1) $\varphi(\delta)$ is continuous and monotonically increasing on $(0, l_0]$; 2) $\varphi(\delta) \neq 0$ and $\lim_{\delta\to 0} \varphi(\delta) = 0$; 3) $\varphi(\delta)/\delta$ is almost decreasing, i.e. $\varphi(\delta_2)/\delta_2 \leqslant C_\varphi \varphi(\delta_1)/\delta_1$ for $\delta_2 > \delta_1$.

Introduce the modulus of continuity of a function $f(t)$, defined on Γ:

$$\omega(f, \delta) = \sup_{|t_1 - t_2| \leqslant \delta} |f(t_1) - f(t_2)|, \qquad 0 < \delta \leqslant l_0.$$

Theorem 1. *If Γ satisfies the conditions stated above and*

$$\omega(f, \delta) \leqslant C_f \varphi(\delta), \qquad \varphi(\delta) \in \Phi, \qquad \int_0^{l/2} \frac{\varphi(\alpha(s))}{\alpha(s)}\, ds < +\infty,$$

then

$$\omega(g, \delta) \leqslant CC_f \left[\int_0^{\beta(\delta)} \frac{\varphi(\alpha(s))}{\alpha(s)}\, ds + \delta \int_{\beta(\delta)}^{l/2} \frac{\varphi(\alpha(s))}{\alpha^2(s)}\, ds \right], \qquad 0 < \delta \leqslant \tilde{l}_0 \leqslant l_0,$$

where

$$g(t_0) = \frac{1}{\pi i} \int_\Gamma \frac{f(t)}{t - t_0}\, dt,$$

and the constant C depends only on the curve Γ and on the constant C_φ ($\tilde{l}_0$ depends only on Γ).

Remark. This theorem* was given in paper (1). Here it is included in order to note the dependence of C only on Γ and C_φ, which will be used essentially in what follows.

With the aid of theorem (1) and the remark, one proves

Theorem 2. *Let Γ be a closed Jordan rectifiable curve, having a tangent at every point and satisfying the condition*

$$S(t_1, t_2) \leqslant K|t_1 - t_2|, \qquad K = \text{const}. \tag{2}$$

Then, if

$$\int_0^{l_0} \frac{\omega(f, \tau)}{\tau}\, d\tau < +\infty,$$

then for the function

$$g(t_0) = \frac{1}{\pi i} \int_\Gamma \frac{f(t)}{t - t_0}\, dt$$

* In paper (1), in inequality (3), in the second integral, $\alpha(s)$ was printed in the denominator; it should read $\alpha^2(s)$.

there is the inequality

$$\omega(g, \delta) \leq C \left[\int_0^\delta \frac{\omega(f, \tau)}{\tau}\, d\tau + \delta \int_\delta^{l_0} \frac{\omega(f, \tau)}{\tau^2}\, d\tau \right], \qquad 0 < \delta \leq l_0, \tag{3}$$

where C depends only on Γ.

An inequality close to inequality (3), in the case of smooth curves, was first obtained by L. G. Magnaradze ([2]).

II. Denote by Ψ the class of positive, continuous functions $\psi(\delta)$, defined on $(0, l_0]$ and having the properties: 1)

$$\int_0^{l_0} \psi(u)\,du = +\infty;$$

2)

$$\int_0^{l_0} u\psi(u)\,du < +\infty.$$

In this section it is assumed that Γ satisfies the conditions of theorem (2).

Let $\psi(\delta) \in \Psi$. Denote by J_ψ the class of functions $f(t)$, defined on Γ, for which

$$\int_0^{l_0} \omega(f,\tau)\psi(\tau)\,d\tau < +\infty.$$

The following theorem partially solves the question of the classification of J_ψ.

Theorem 3. Let $\psi_1(\delta), \psi_2(\delta) \in \Psi$. If

$$0 < \underline{\lim}_{\delta\to 0}(\psi_1(\delta)/\psi_2(\delta)) \leq \overline{\lim}_{\delta\to 0}(\psi_1(\delta)/\psi_2(\delta)) < +\infty,$$

then J_{ψ_1} and J_{ψ_2} coincide, while if

$$\lim_{\delta\to 0}(\psi_1(\delta)/\psi_2(\delta)) = +\infty,$$

then J_{ψ_1} is a proper part of J_{ψ_2}.

With the aid of theorems 2 and 3 one proves

Theorem 4. Let $\psi(\delta) \in \Psi$ and

$$\lim_{\delta\to 0} \delta^2\psi(\delta) = 0, \qquad \lim_{\delta\to 0}\left(\delta\psi(\delta)\Big/\int_\delta^{l_0} \psi(\tau)\,d\tau\right) = K \quad (0 < K < 1).$$

Then J_ψ is invariant with respect to the operator

$$Af = \frac{1}{\pi i}\int_\Gamma \frac{f(t)}{t - t_0}\,dt.$$

It is not difficult to verify that the functions $\psi(\delta) = 1/\delta^{1+\varepsilon}$, $\psi(\delta) = \ln|1/\delta|/\delta^{1+\varepsilon}$ $(0 < \varepsilon < 1)$ satisfy the conditions of theorem 4, and the classes J_ψ generated by them, by virtue of theorem 3, are different for different ε.

For the further arguments the following is useful

Lemma. Let $\psi(\delta) \in \Psi$ and

$$\lim_{\delta\to 0} \delta^2\psi(\delta) = 0, \qquad \lim_{\delta\to 0} \left(\delta\psi(\delta) \Big/ \int_\delta^{l_0} \psi(\tau)\, d\tau \right) = 0. \tag{4}$$

Then the function

$$\frac{1}{\delta} \int_\delta^{l_0} \psi(\tau)\, d\tau = \psi_1(\delta) \in \Psi$$

also satisfies conditions (4).

With the aid of theorems 2 and 3 and the lemma one proves

Theorem 5. Let $\psi(\delta) \in \Psi$ and satisfy conditions (4). Then the operator A maps $J_{\psi_{i+1}}$ into J_{ψ_i}, where

$$\psi_{i+1}(\delta) = \frac{1}{\delta} \int_\delta^{l_0} \psi_i(\tau)\, d\tau \quad (i = 0, 1, 2, \ldots), \qquad \psi_0(\delta) = \psi(\delta),$$

and $J_{\psi_{i+1}}$ is a proper part of J_{ψ_i}.

This result in the case $\psi(\delta) = 1/\delta$ was obtained by L. G. Magnaradze ([2]). Theorem 5 makes it possible to construct a sequence $\{J_{\psi_i}\}$ different from the sequence $\{J_i\}$ constructed by L. G. Magnaradze in the same paper.*

We note that, by virtue of Zygmund' s estimate ([3]), Theorems 4 and 5 are also valid for trigonometrically conjugate functions.

Remark. Let us point out that X. Sjoe-Mou ([4]) succeeded in proving analogues of the theorems of L. G. Magnaradze expressing the relation between the modulus of continuity of $\varphi(t)$ in L_p $(p > 1)$ and the modulus of continuity in L_p of the angular boundary values of the Cauchy-type integral

$$F(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\varphi(t)}{t - z}\, dt$$

(Γ is a closed Jordan rectifiable curve satisfying condition (2)).

By virtue of the estimate, obtained in the same paper, expressing the relation between the modulus of continuity of $\varphi(t)$ in L_p and the modulus of continuity in L_p of the angular boundary values $F(z)$, Theorems 4 and 5 are valid in this case as well.

III. Consider the singular integral

$$\Phi(t_0, \tau) = \int_\Gamma \frac{f(t, \tau)}{t - t_0}\, dt,$$

where $t_0 \in \Gamma,\ \tau \in D$ (D is some bounded set in the complex plane). Denote

$$\omega_t(f, \delta) = \sup_\tau \sup_{|t_1 - t_2| \le \delta} |f(t_1, \tau) - f(t_2, \tau)|\,,$$

$$\omega_\tau(f, \delta) = \sup_t \sup_{|\tau_1 - \tau_2| \le \delta} |f(t, \tau_1) - f(t, \tau_2)|\,.$$

With the aid of Theorem 1 one proves

Theorem 6. Let Γ be a closed Jordan rectifiable curve, having a tangent at every point and satisfying condition (1). If

$$\omega_t(f, \delta) = O[\varphi(\delta)], \qquad \omega_\tau(f, \delta) = O[\widetilde{\varphi}(\delta)],$$

$$\varphi(\delta) \in \Psi[\beta(\delta)] \cap \Psi_1[\beta(\delta)]^{**}, \qquad \widetilde{\varphi}(\delta) \int_{\beta(\delta)}^{1/2} \frac{ds}{\alpha(s)} = O\left[\varphi(\delta)\frac{\beta(\delta)}{\delta}\right]$$

($\widetilde{\varphi}(\delta)$ is a positive function), then

$$\omega_{t_0}(\Phi, \delta) = O\left[\varphi(\delta)\frac{\beta(\delta)}{\delta}\right], \qquad \omega_\tau(\Phi, \delta) = O\left[\varphi(\delta)\frac{\beta(\delta)}{\delta}\right].$$

Let us note one important particular case of this theorem, which is a generalization of a theorem of N. I. Muskhelishvili ([5]) on a singular integral containing a parameter.

Theorem 7. Let Γ be a closed Jordan rectifiable curve, having a tangent at every point and

$$S(t_1, t_2) \le \text{const}\, |t_1 - t_2|^\gamma \qquad (0 < \gamma \le 1).$$

If

$$\omega_t(f, \delta) = O[\delta^\alpha], \qquad \omega_\tau(f, \delta) = O[\delta^{\alpha_1}], \qquad 1 - \gamma < \alpha < \alpha_1 \le 1,$$

then

$$\omega_{t_0}(\Phi, \delta) = O[\delta^{\alpha - (1-\gamma)}], \qquad \omega_\tau(\Phi, \delta) = O[\delta^{\alpha - (1-\gamma)}].$$

IV. Denote $\bigcup_{\alpha>\beta} H_\alpha$ by M_β $(0 \leq \beta < 1)$, where H_α is the class of functions,

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* For example, it follows from Theorem 3 that, if $\psi(\delta) = |\ln|\ln(1/\delta)||/\delta$, then for every $i = 1, 2, \ldots$ the class J_{ψ_i} will be strictly contained between the classes J_{i+1} and J_i $(J_{i+1} \subset J_{\psi_i} \subset J_i)$.

** The definition of the classes $\Psi[\beta(\delta)]$ and $\Psi_1[\beta(\delta)]$ is given in ([1]).

satisfying on Γ the Hölder condition with exponent α. Consider the singular integral equation

$$R\varphi = A(t_0)\varphi(t_0) + \frac{B(t_0)}{\pi i} \int_\Gamma \frac{\varphi(t)}{t - t_0}\, dt + \frac{1}{\pi i} \int_\Gamma N(t_0, t)\varphi(t)\, dt = f(t_0)$$

and its adjoint equation

$$R'\psi = A(t_0)\psi(t_0) - \frac{1}{\pi i} \int_\Gamma \frac{B(t)\psi(t)}{t - t_0}\, dt + \frac{1}{\pi i} \int_\Gamma N(t, t_0)\psi(t)\, dt = g(t_0).$$

With the aid of Theorem 7 and the Carleman–Vekua method [(6)], the following is proved.

Theorem 8. Let Γ satisfy the conditions of Theorem 7, $2/3 < \gamma \leq 1$. If $A^2(t_0) - B^2(t_0) \neq 0$, $t_0 \in \Gamma$, $A(t_0), B(t_0) \in M_{3(1-\gamma)}$; $N(t_0, t)$, in both arguments, uniformly respectively in t and t_0, belongs to $M_{2(1-\gamma)}$, then the following assertions are true:

1. The number of linearly independent solutions of the equations $R\varphi = 0$ and $R'\psi = 0$ in $M_{1-\gamma}$ is finite.
2. For $f(t_0) \in M_{2(1-\gamma)}$, in order for the equation $R\varphi = f$ to be solvable in $M_{1-\gamma}$, it is necessary and sufficient that

$$\int_\Gamma f(t)\psi_k(t)\, dt = 0 \qquad (k = 1, \ldots, m'),$$

where $\psi_1(t), \ldots, \psi_{m'}(t)$ is a complete system of linearly independent solutions of the adjoint homogeneous equation $R'\psi = 0$ in $M_{1-\gamma}$.

3. If by m and m' we denote respectively the number of linearly independent solutions of $R\varphi = 0$ and $R'\psi = 0$ in $M_{1-\gamma}$, then

$$m - m' = \frac{1}{2\pi i} \left[\ln \frac{A - B}{A + B}\right]_\Gamma,$$

where $[\ ]_\Gamma$ denotes the increment of the expression in brackets when traversing Γ in the positive direction.

Azerbaijan State University
named after S. M. Kirov

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Note: Figure translations are in progress. See original paper for figures.