OCEANIC CIRCULATION AND THE EKMAN PROBLEM

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OCEANIC CIRCULATION AND THE EKMAN PROBLEM

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After the publication of the works of H. Stommel ([1]) and W. Munk ([2]), the opinion took shape that in the classical theory of marine currents, proceeding from the equations of the steady motion of a homogeneous fluid and boundary conditions in Ekman' s form ([3,4]), it is impossible to obtain the effect of western intensification of ocean currents, caused by the nonuniformity of the wind and by the latitudinal variation of the Coriolis parameter. At the same time, the study of this important feature of oceanic circulation from the point of view of the classical theory is of interest because in this theory not only the vertically averaged current velocity ([1]), or the total transport ([2]), is determined, but also the current velocity at any horizon within the water layer that is assumed to be encompassed by the wind-driven current. Only comparatively recently ([5]), within the framework of the indicated theory, was it possible to obtain an equation for the integral stream function, making it possible in principle to obtain all the main features of oceanic circulation studied by Stommel and Munk. Moreover—and this is the main point—the possibility opened up of studying new effects connected, in particular, with the vertical structure of currents.

The basic relations of the theory have the form ([6])

$$\vartheta' \Delta\psi + (\partial\vartheta'/\partial x + \partial\lambda'/\partial y)\, \partial\psi/\partial x + (\partial\vartheta'/\partial y - \partial\lambda'/\partial x)\, \partial\psi/\partial y = \mathrm{rot}_z\, m'T + \mathrm{div}\, n'T; \tag{1}$$

$$\partial\zeta/\partial x = -m'T_x + n'T_y - \lambda'\partial\psi/\partial x - \vartheta'\partial\psi/\partial y, \tag{2}$$

$$\partial\zeta/\partial y = -n'T_x - m'T_y + \vartheta'\partial\psi/\partial x - \lambda'\partial\psi/\partial y;$$

$$u = NT_x + MT_y + \Theta\, \partial\zeta/\partial x + \Lambda\, \partial\zeta/\partial y,$$

$$v = -MT_x + NT_y - \Lambda\,\partial\zeta/\partial x + \Theta\,\partial\zeta/\partial y, \tag{3}$$

where u, v are the horizontal components of the current velocity along the Cartesian coordinate axes X, Y, Z, directed respectively along the equator to the east, to the north, and vertically downward (the origin is at the undisturbed sea surface); ζ is the level; T is the tangential wind stress; $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the Laplace operator. Finally, ψ is the integral stream function, connected with the components of the total transport by the relations:

$$s_x = \int_\zeta^H u\,dz = -\partial\psi/\partial y; \qquad s_y = \int_\zeta^H v\,dz = \partial\psi/\partial x.$$

All the coefficients in (1)–(3) can be expressed through certain dimensionless standard functions ([6]). If the oceanic region includes the equator, in whose vicinity computation of the standard functions is difficult, then it is expedient to introduce throughout the region new standard functions

$$f_i^* = f_i(aH)^{-2}, \qquad F_i^* = F_i(aH)^{-2}, \qquad f_j^* = f_j(aH)^2,$$

where $i = 1, 2, 3, 4$ and $j = 5, 6$. The relation of the coefficients to the standard functions is as follows:

$$\lambda' = \frac{\Omega}{gH}(aH)^{-2} f_5^*, \qquad \vartheta' = \frac{\Omega}{gH}(aH)^{-2} f_6^*,$$

$$m' = \frac{1}{g\rho H}(f_1^* f_5^* + f_2^* f_6^*), \qquad n' = \frac{1}{g\rho H}(f_2^* f_5^* - f_1^* f_6^*),$$

$$N = \frac{1}{g\rho\Omega}(aH)^2 F_1^*, \qquad M = \frac{1}{g\rho\Omega}(aH)^2 F_2^*,$$

$$\Theta = \frac{g}{\Omega}(aH)^2 F_3^*, \qquad \Lambda = \frac{g}{\Omega}(aH)^2 F_4^*.$$

We do not introduce the functions f_9^* and f_{10}^*, since the functions F^* are more conveniently determined directly from the formulas

$$F_1^* = (2/aH)[(\operatorname{ch} aH \cos aH + \operatorname{sh} aH \sin aH)\operatorname{ch}\eta \sin\eta +$$

$$+(\operatorname{ch} aH \cos aH - \operatorname{sh} aH \sin aH)\operatorname{sh}\eta\cos\eta]/(\operatorname{ch} 2aH + \cos 2aH),$$

$$F_2^* = (2/aH)[(\operatorname{ch} aH \cos aH + \operatorname{sh} aH \sin aH)\operatorname{sh}\eta\cos\eta -$$

Fig. 1

Figure 1: Fig. 1

$$-(\operatorname{ch} aH \cos aH - \operatorname{sh} aH \sin aH) \operatorname{ch} \eta \sin \eta] / (\operatorname{ch} 2aH + \cos 2aH),$$

$$F_3^* = (\operatorname{sh} \vartheta \sin \eta + \operatorname{sh} \eta \sin \vartheta) / (aH)^2 (\operatorname{ch} 2aH + \cos 2aH),$$

$$F_4^* = (\operatorname{ch} 2aH + \cos 2aH - \operatorname{ch} \vartheta \ \operatorname{ch} \eta +$$

$$+ \operatorname{ch} \eta \cos \vartheta) / (aH)^2 (\operatorname{ch} 2aH + \cos 2aH).$$

Already not far from the equator, practically for $aH > 1$, the new standard functions are easily computed directly, as are the old standard functions. In the vicinity of the equator, however, they are computed in series, which converge rapidly. For $aH < 1$ the functions f_5^* and f_6^*, for example, can be computed with quite sufficient accuracy from the formulas

Fig. 1

$$f_5^* = \left[{}^6/_5 (aH)^2 + {}^{256}/_{315} (aH)^6\right] / \left[1 + {}^{356}/_{525} (aH)^4\right],$$

$$f_6^* = \left[{}^3/_2 + {}^{36}/_{35} (aH)^4 + {}^{496}/_{3465} (aH)^8\right] / \left[1 + {}^{358}/_{315} (aH)^4 + {}^{1840}/_{6615} (aH)^8\right].$$

The results of computations of the functions f_5^*, f_6^* for $0 \leq aH \leq 1$ are shown graphically in Fig. 1.

The standard functions are determined most simply at the equator itself ($aH = 0$), where $f_1^* = f_3^* = f_5^* = F_2^* = F_4^* = 0$; $f_2^* = 1$, $f_4^* = {}^1/_3$, $f_6^* = {}^3/_2$, $F_1^* = 2(1 - z')$, $F_3^* = 1 - {z'}^2$. From (2), (3) we obtain for the equator

$$u = \frac{H - z}{\rho A} T_x + \frac{g}{A} \frac{H^2 - z^2}{2} \frac{\partial \zeta}{\partial x}, \qquad v = \frac{H - z}{\rho A} T_y + \frac{g}{A} \frac{H^2 - z^2}{2} \frac{\partial \zeta}{\partial y}; \tag{4}$$

$$\frac{\partial \zeta}{\partial x} = -\frac{3}{2 g \rho H} T_x - \frac{3A}{g H^3} \frac{\partial \psi}{\partial y}, \qquad \frac{\partial \zeta}{\partial y} = -\frac{3}{2 g \rho H} T_y + \frac{3A}{g H^3} \frac{\partial \psi}{\partial x}. \tag{5}$$

Thus, from the standpoint of the relation between the flow velocity, the level slope, and the integral stream function, at the equator we have an analogue of the shallow sea [6].

Fig. 2

Figure 2: Fig. 2

Let us now introduce the dimensionless parameter

$$f_0 = \frac{\Omega \rho k}{\sqrt{2}\gamma} \frac{H}{W}, \tag{6}$$

in which W is the modulus of the wind velocity, k is the mean statistical value of the wind coefficient, and γ is the proportionality coefficient in the quadratic dependence of the tangential wind stress on its velocity ([6]). The parameter f_0, on the other hand, is related to the dimensionless argument aH by the relation

$$f_0 = \frac{\sqrt{2}}{4}(aH)^2 s^*, \tag{7}$$

where

$$s^* = \frac{2\sqrt{2}}{aH}\sqrt{s}.$$

At the equator itself $s^* = 1$, and far from it ($aH \to \infty$) $s^* = 0$. Near the equator s^* is easily computed in series. The results of the calculations are presented graphically in Fig. 2.

In studying currents in seas we previously assumed the coefficient k to be constant ([6]). When considering oceanic currents, the dependence of this coefficient on the latitude of the place can be taken into account, using the empirical formula

$$k = 0.0127/\sqrt{\sin(\varphi + \varphi_0)}, \tag{8}$$

in which φ_0 is a certain constant quantity, determined as a function of the mean statistical value of the wind coefficient at the equator itself. This value, according to available observations in the Atlantic Ocean, may be taken as equal to 0.2. Let us note that such a large value of the coefficient k is apparently connected with the peculiarity of the dynamic processes in the equatorial region, in particular with the development of the jump layer, as a result of which the vertical exchange of momentum is impeded and the coefficient of vertical exchange decreases as the equator is approached.

Fig. 2

For $k = 0.2$ we obtain a very small value $\varphi_0 = 0.004$, so that formula (8) not far from the equator practically passes into Ekman' s formula ([3,4]), which follows from formula (8) for $\varphi_0 = 0$.

Having calculated from formula (6) the value of the parameter f_0 as a function of the ratio of the depth H to the wind velocity W, the latitude of the place φ ($\Omega = 2\omega \sin\varphi$), and the coefficient k, we can then, with the aid of relation (7), compute the dimensionless argument of the standard functions aH. After this, all coefficients in the basic relations of the theory (1), (2), (3) are determined. Integrating equation (1) under the boundary condition

$$(\psi)_L = 0, \tag{9}$$

where L is the contour of the closed basin, we obtain the solution of the problem.

Passing to the boundary problems associated with the integration of the basic equation (1), let us first consider the case of an oceanic region situated far from the equator. In this case $2aH \gg 1$, and instead of (1), with sufficient accuracy, we obtain the equation

$$\frac{\Omega}{2aH}\Delta\psi + \frac{d\Omega}{dy}\frac{\partial\psi}{\partial x} - \frac{\Omega}{H}\frac{\partial H}{\partial y}\frac{\partial\psi}{\partial x} = \frac{H}{\rho}\operatorname{rot}_z \frac{\mathbf{T}}{H}. \tag{10}$$

If, in addition, the depth $H = \text{const}$, then equation (10) simplifies:

$$\frac{\Omega}{2aH}\Delta\psi + \frac{d\Omega}{dy}\frac{\partial\psi}{\partial x} = \frac{1}{\rho}\operatorname{rot}_z \mathbf{T}. \tag{11}$$

Equation (11), to within the coefficients, coincides with Stommel' s equation ([1,5])*.

For an arbitrary contour L, the solution of equations (1), (10), or (11) is found by a numerical method with the aid of a computer. In some special cases an analytical solution can also be found. In particular, it can be found for a rectangular basin (at constant depth) located far from the equator, with a wind varying according to the law $W = W_0 \cos\varphi \cdot (\sin\varphi)^{-1/2}$. In this case, instead of equation (11), we shall have an equation with constant coefficients

$$C\Delta\psi + \frac{\partial\psi}{\partial x} = \frac{1}{\beta\rho}\operatorname{rot}_z T \tag{12}$$

(C is a certain constant, β is the latitudinal variation of the Coriolis parameter), which is integrated in the same way as Stommel' s equation.

An analytical solution can also be found for a zonal channel, closed at infinity, with a wind varying only in the transverse direction (7). In this case all functions are independent of the coordinate x, and from (1) we obtain

$$\frac{d}{dy}\left(\vartheta'\frac{d\psi}{dy}\right) = -\frac{dm'T_x}{dy} + \frac{dn'T_y}{dy}, \tag{13}$$

whence

$$s_x = -\frac{\partial\psi}{\partial y} = \frac{1}{\vartheta'}[m'T_x - n'T_y + c]. \tag{14}$$

The constant of integration c is determined from the condition of closure of the channel

$$\int_{-b_1}^{b_2} s_{xdy} = 0, \tag{15}$$

where $b_2 - b_1$ is the transverse dimension of the channel. After simple transformations we obtain

$$s_x = \frac{1}{\vartheta'}(m'T_x - n'T_y) - \frac{\overline{(m'T_x/\vartheta')} - \overline{(n'T_y/\vartheta')}}{\vartheta'\overline{(1/\vartheta')}}. \tag{16}$$

It is evident from formula (16) that the total flux in the transverse section of the channel differs from zero even in the case when the wind is uniform. This conclusion, as is evident from (1), remains valid also for the case of an arbitrary contour of a closed basin at constant values of the depth H and of the coefficient of vertical exchange A.

Thus, equation (1) is more general than the equation of Stommel ([1]) or of Munk ([2]). Moreover, by solving this equation, as was already indicated, we are able to determine the current velocity at any horizon. Finally, the use of equation (1) makes it possible to consider problems connected with the peculiarities of currents near the equator ([8]), whereas these peculiarities cannot be obtained from the equations of Stommel and Munk.

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* Complete coincidence can be obtained by replacing, in equation (11), the variable coefficient at the higher derivatives by its mean value and by choosing in the corresponding way the value of the coefficient of vertical exchange.

Note: Figure translations are in progress. See original paper for figures.

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