ASYMPTOTIC REPRESENTATIONS OF SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS IN THE CASE OF MULTIPLE ELEMENTARY DIVISORS OF THE CHARACTERISTIC EQUATION

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Abstract

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ASYMPTOTIC REPRESENTATIONS OF SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS IN THE CASE OF MULTIPLE ELEMENTARY DIVISORS OF THE CHARACTERISTIC EQUATION

1. Consider a system of first rank containing a real large parameter λ:

$$\dot{x} = A(\lambda, t)x, \tag{1}$$

where the matrix $A(\lambda, t)$ has the form

$$A(\lambda, t) = \lambda A^{(0)} + A_1 + \lambda^{-1}A^{(2)} + ... + \lambda^{-k}A^{(k+1)}(t, \lambda).$$

The equation

$$\left|A^{(0)} - \mu^* E\right| = 0 \tag{2}$$

is called the **characteristic** equation.

Ya. D. Tamarkin was apparently the first to observe that, in the case of multiple elementary divisors, the asymptotic representations of the integrals of system (1) may contain fractional powers of the parameter λ, and the very structure of these representations is determined not only by the matrix $A^{(0)}$, but also by the properties of the matrices $A^{(i)}$. This note contains an exposition of a method that in some cases makes it possible to construct effectively asymptotic representations of particular solutions of system (1) in the case where the matrix $\left\|A^{(0)} - \mu^* E\right\|$ has non-simple elementary divisors.

2. Consider the system

$$\dot{x}_1 = \lambda\mu(t)x_1 + \lambda x_2 + \sum_j a_{1j}(t)x_j,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\dot{x}_{m-1} = \lambda\mu(t)x_{m-1} + \lambda x_m + \sum_j a_{m-1,j}(t)x_j, \tag{3}$$

$$\dot{x}_m = \lambda\mu(t)x_m + \sum_j a_{mj}(t)x_j.$$

System (3) has one elementary divisor $\mu - \mu^*$, whose multiplicity is equal to m. Put

$$x_k = u_k(\lambda, t)\exp\int_0^t [\lambda\mu(t) + \varphi(t,\lambda)]\,dt. \tag{4}$$

Then

$$\dot{u}_1 + u_1\varphi = \lambda u_2 + \sum_j a_{1j}u_j,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\dot{u}_{m-1} + u_{m-1}\varphi = \lambda u_m + \sum_j a_{m-1,j}u_j, \tag{5}$$

$$\dot{u}_m + u_m\varphi = \sum_j a_{mj}u_j.$$

Denote by k_i and k the highest powers of the parameter λ in the representations of the functions u_i and φ. We shall also assume that $a_{m1} \neq 0$ for all $t \in [0, T]$.

Consider the system of equations

$$\begin{gathered} k_1 - k_2 = 1 - k, \\ k_2 - k_3 = 1 - k, \\ \dots \\ k_{m-1} - k_m = 1 - k, \\ k_m - k_1 = -k \end{gathered} \tag{6}$$

with respect to the quantities $k_1, k_2, \dots, k_m$. Its determinant is equal to zero. In order for it to have solutions, it is necessary and sufficient that the rank of the augmented matrix also be equal to zero. This condition makes it possible to determine k:

$$k = (m-1)/m. \tag{7}$$

Without loss of generality, put $k_1 = 0$. Then

$$k_2 = -1/m, \quad k_3 = -2/m, \ \dots, \quad k_m = -(m-1)/m.$$

Put

$$\begin{gathered} u_i(t,\lambda) = \lambda^{k_i} u_{i0}(t) + \lambda^{k_i - 1/m} u_{i1}(t) + \lambda^{k_i - 2/m} u_{i2}(t) + \dots, \\ \varphi(t,\lambda) = \lambda^k \varphi_0(t) + \lambda^{k-1/m}\varphi_1(t) + \dots + \lambda^{1/m}\varphi_{m-2}(t) + \dots, \\ i = 1, 2, \dots, m. \end{gathered} \tag{8}$$

Substituting the series (8) into the system (5) and comparing the coefficients of the highest powers of the parameter λ, we obtain

$$\begin{gathered} u_{10}\varphi_0 = u_{20}, \\ u_{20}\varphi_0 = u_{30}, \\ \dots \\ u_{m0}\varphi_0 = a_{m1}u_{10}. \end{gathered} \tag{9}$$

In order for the system (9) to have nontrivial solutions, it is necessary and sufficient that φ_0 be a root of the equation

$$\begin{vmatrix} \varphi_0 & -1 & 0 \dots 0 & \\ 0 & \varphi_0 & -1 \dots 0 & \\ \dots & \dots & \dots & \dots \\ -a_{m1} & 0 & 0 \dots \varphi_0 & \end{vmatrix} = 0,$$

whence

$$\varphi_0 = \sqrt[m]{a_{m1}}. \tag{10}$$

Denote by $\varphi_0^{(s)}$ $(s = 1, 2, \dots, m)$ any one of the roots of the resolving equation (10), and take $\varphi_0 = \varphi_0^{(s)}$. Leaving arbitrary the function u_{i0}, for example u_{10}, we express the remaining ones as

$$u_{i0} = \varphi_0^{i-1} u_{10}.$$

The functions u_{i1} will satisfy the system of equations

$$\begin{gathered} u_{11}\varphi_0 - u_{21} = -u_{10}\varphi_1, \\ u_{21}\varphi_0 - u_{31} = -u_{20}\varphi_1, \\ \dots \\ u_{m1}\varphi_0 - a_{m1}u_{11} = -u_{m0}\varphi_1 + a_{m2}u_{20}. \end{gathered} \tag{11}$$

The determinant of system (11) is equal to zero. The condition for its solvability gives a linear equation with respect to φ_1

$$\varphi_1 = (a_{m-1,1} + a_{m2})/m\varphi_0^{m-2}$$

and, analogously,

$$\varphi_j = (a_{m-j,1} + a_{m-j+1,2} + \dots + a_{m,j+1})/m\varphi_0^{m-j-1}, \quad j = 1, 2, \dots, m-2; \tag{12}$$

$$u_{ij} = \varphi_0^{i-1}u_{1j}, \quad i = 1, 2, \dots, m; \quad j = 1, 2, \dots \tag{13}$$

The function u_{1j} remains undetermined at this step.

The function u_{10} is determined from the solvability condition of the system written for determining the functions $u_{i,m-1}$:

$$Lu_{10} \equiv \dot{u}_{10} - u_{10}[a_{11} + a_{22} + \dots + \dot{\varphi}_0/2\varphi_0] \tag{14}$$

or

$$u_{10} = \frac{C}{\sqrt{\varphi_0}} \exp \int_0^t (a_{11} + \dots + a_{mm})\, dt,$$

where C is an arbitrary constant. And so on: the functions u_{ir} are determined from the solvability conditions of the system of equations with respect to $u_{i,m+r-1}$, which are found from formulas (13).

3. Having determined N terms of the expansion (8), we can construct the expressions

$$x_{iN}^{(s)} = \lambda^{k_i}\{u_{i0}^{(s)} + \lambda^{-1/m}u_{i1}^{(s)} + \dots\} \exp \int_0^t [\lambda\mu + \lambda^k\varphi_0 + \dots + \lambda^{1/m}\varphi_{m-2}]\, dt, \tag{15}$$

where $\varphi_j^{(s)}$ and $u_{ij}^{(s)}$ denote the functions corresponding to the root of equation (10) with number s. The matrix

$$\left\| x_{iN}^{(s)} \right\| \tag{16}$$

may be taken as an approximate representation of the matrix of fundamental solutions of system (3). The basis for this is the following

Theorem. *Let $a_{m1} \neq 0$ for any $t \in [0, T]$; then matrix (16) gives a uniform, on $[0, T]$, approximation of the matrix of fundamental solutions of system (3) in the following sense: if a particular solution $x_i^*(t)$ of system (3) and the function $x_{iN}^{(s)}(t)$ satisfy the same initial conditions, then for any $t \in [0, T]$*

$$|x_i^*(t) - x_{iN}^{(s)}(t)| = O(\lambda^{k_i - N/m}). \tag{17}$$

This result can be extended to the general case. If $a_{m1} \neq 0$, then to each elementary divisor of multiplicity m there corresponds a group of m solutions of the form (15).

If $a_{m1} \equiv 0,\ a_{m2} \equiv 0, \dots, a_{ml} \neq 0$, then the asymptotics is constructed in the same way, but in this case

$$k = (m - l)/(m - l + 1), \qquad \varphi_0 = \sqrt[m-l+1]{a_{ml}}.$$

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CITED LITERATURE

1. Ya. D. Tamarkin, *On Certain General Problems of the Theory of Ordinary Linear Differential Equations and on the Expansion of Arbitrary Functions in Series*, Petrograd, 1917.

Note: Figure translations are in progress. See original paper for figures.