ON THE QUESTION OF THE DIRECTIONAL STABILITY OF VEHICLES ON PNEUMATIC WHEELS

MECHANICS

1966

SovietRxiv

Abstract

Full Text

UDC 531

MECHANICS

Yu. I. NEIMARK, N. A. FUFAEV

ON THE QUESTION OF THE DIRECTIONAL STABILITY OF VEHICLES ON PNEUMATIC WHEELS

(Presented by Academician A. Yu. Ishlinskii, 23 XII 1965)

1. The practical importance of the problem of the directional stability of a motorcycle, an automobile, and aircraft landing gear has led to the creation of a number of theories of the rolling of a pneumatic wheel: from the widely disseminated drift hypothesis to the most complete theory of the rolling of an elastic pneumatic tire, developed by M. V. Keldysh ([1−12]). The theories created up to now are not interconnected and reflect different approaches to the study of the influence of the deformability of the pneumatic tire on the process of its rolling.

In the present work it is shown that a simple analysis of the equations obtained on the basis of M. V. Keldysh' s theory makes it possible not only to establish a connection between the various theories, but also, essentially, to arrive at a generalization of the drift hypothesis and to indicate the conditions of its applicability. The proposed generalizations of the drift hypothesis can substantially facilitate the investigation of stability, because, in comparison with the theory of M. V. Keldysh, in a system containing m independent pneumatic wheels there occurs a reduction of the order of the characteristic equation by m or $2m$.

2. Following the theory of M. V. Keldysh ([3]), let us compose the equations of motion of a vehicle with m pneumatic wheels for its small deviations from rectilinear motion along the axis OY with constant speed V. Denote by $q_1, q_2, \dots, q_n$ the generalized coordinates of the vehicle and, in addition, introduce quantities characterizing the position of the i-th wheel ($i = 1, 2, \dots, m$) and the deformation of its pneumatic tire. Let x_i be the coordinate of the point K_i where the straight line of greatest slope, passing in the mean plane of the wheel through its center, meets the plane XOY of the road; χ_i the angle between the perpendicular to the road and the mean plane of the wheel; θ_i the angle between the axis OY and the track of the mean plane of the wheel on the road; ξ the lateral displacement of the center of the contact patch of the pneumatic tire relative to the point K_i; φ_i the angle of twisting of the contact patch with respect to the wheel rim. The introduced quantities x_i, θ_i, χ_i are, obviously, certain functions of the generalized coordinates $q_1, q_2, \dots, q_n$. Suppose that there is no slipping of the pneumatic tires, and that the quantities $\xi_i, \varphi_i, \theta_i, \chi_i, q_j$ $(i = 1, 2, \dots, m;\ j = 1, 2, \dots, n)$ are small, i.e., small deformation of the pneumatic tires is considered for small deviations of the vehicle from its steady motion. Under these assumptions, the reaction forces from the road acting on the wheels and causing deformation of the pneumatic tires can be related to the magnitude of the deformation by a linear law, and the potential energy U of the deformed pneumatic tires can be expressed in the form

$$U = \frac{1}{2} \sum_{i=1}^{m} \left(a_i \xi_i^2 + 2\sigma_{iN} i \xi_i \chi_i + \rho_{iN} i \chi_i^2 + b_i \varphi_i^2 \right), \tag{1}$$

where $a_i, b_i, \sigma_i, \rho_i$ are constants for a given normal load N_i on the i-th pneumatic wheel. Let $T(q, \dot{q})$ be the kinetic energy of the vehicle; Q_j $(j = 1, 2, \dots, n)$ the generalized forces, in whose calculation are taken into account

all forces except the forces of deformation of the pneumatic tires. When an elastic pneumatic tire rolls without slipping, kinematic constraints are imposed on the system, represented by the equations

$$\dot{x}_i + \dot{\xi}_i + V\theta_i + V\varphi_i = 0; \qquad \dot{\theta}_i + \dot{\varphi}_i - \alpha_i V \xi_i + \beta_i V \varphi_i + \gamma_i V \chi_i = 0, \tag{2}$$

$$i = (1, 2, \dots, m),$$

where $\alpha_i, \beta_i, \gamma_i$ are constants. Thus, we arrive at the problem of analytical mechanics of nonholonomic systems. Using the equations of dynamics in Maggi' s form (13), we obtain the equations*

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j + \sum_{i=1}^{m} \left(\frac{\partial U}{\partial \xi_i}\frac{\partial x_i}{\partial q_j} - \frac{\partial U}{\partial \chi_i}\frac{\partial \chi_i}{\partial q_j} + \frac{\partial U}{\partial \varphi_i}\frac{\partial \theta_i}{\partial q_j} \right), \tag{3}$$

$$(j = 1, 2, \dots, n),$$

which, together with (2), represent the equations of motion of a vehicle on pneumatic wheels. The system of equations (2), (3) describes the motion of the representative point in a $2(n+m)$-dimensional phase space $(q, \dot{q}, \xi, \varphi)$. The state of equilibrium in this space corresponds to the steady motion of the vehicle, whose stability is determined by the roots of a characteristic equation of order $2(n+m)$.

3. The equations of motion (2), (3) can be simplified in cases of a sufficiently large velocity V, or sufficiently large values of the kinematic parameters α_i, β_i, and γ_i.

Under the first assumption, introducing the small parameter $\mu = 1/V$, we write the group of equations (2) in the form

$$\mu\dot{\xi}_i = -dx_i/ds - \theta_i - \varphi_i, \qquad \mu\dot{\varphi}_i = -d\theta_i/ds + \alpha_i\xi_i - \beta_i\varphi_i - \gamma_i\chi_i$$

$$(i = 1, 2, \dots, m).$$

Here $s = Vt$ is the distance traveled, while the quantities dx_i/ds—the slip angle —and $d\theta_i/ds$—the curvature of the rolling line of the i-th wheel—are functions of the generalized coordinates $q_1, q_2, \dots, q_n$. In the case under consideration, equations (2) prove to be differential equations with a small parameter multiplying the derivative.

As is known (14), as $\mu \to 0$ a surface G of slow motions appears in the phase space $(q, \dot{q}, \xi, \varphi)$, determined by the equations

$$dx_i/ds + \theta_i + \varphi_i = 0, \qquad d\theta_i/ds - \alpha_i\xi_i + \beta_i\varphi_i + \gamma_i\chi_i = 0$$

$$(i = 1, 2, \dots, m), \tag{4}$$

from which the relation follows

$$\dot{x}_i + \frac{1}{\beta_i}\dot{\theta}_i = -\frac{\alpha_i}{\beta_i}V\xi_i + \frac{\gamma_i}{\beta_i}V\chi_i - V\theta_i. \tag{5}$$

According to the characteristic equation

$$\prod_{i=1}^{m}(p^2 + \beta_i V p + \alpha_i V^2) = 0, \tag{6}$$

all of whose roots have negative real parts, the surface of slow motions G is stable with respect to fast motions. Consequently, for any initial conditions the representative point in the phase space $(q, \dot{q}, \xi, \varphi)$ "jumps" onto the surface G, where it continues to move in accordance with equations (3), (4).

---

* It can be shown that the constraints (2) and the condition $y = Vt$ lead to possible displacements satisfying the conditions $\delta y = 0$, $\delta\xi_i + \delta x_i = 0$, and $\delta\varphi_i + \delta\theta_i = 0$, on which the work of the constraint reaction forces is zero.

Thus, in the case of sufficiently high velocities, the motion of the vehicle can be regarded as "slow" for the variables ξ_i, φ_i, which characterize the deformation of the pneumatic tire. Eliminating these variables by means of relations (4) and using expression (1), we obtain

$$\frac{\partial U}{\partial \xi_i} = \frac{a_{1i}}{V}\dot{\theta}_i - a_{2i}\theta_i - \frac{a_{2i}}{V}\dot{x}_i + a_{3i}\chi_i \equiv F_i,$$

$$\frac{\partial U}{\partial \chi_i} = \frac{b_{1i}}{V}\dot{\theta}_i - b_{2i}\theta_i - \frac{b_{2i}}{V}\dot{x}_i + b_{3i}\chi_i \equiv -M_{xi}, \tag{7}$$

$$\frac{\partial U}{\partial \varphi_i} = -\frac{b_i}{V}\dot{x}_i - b_i\theta_i \equiv M_{\theta i} \qquad (i = 1, 2, \dots, m).$$

The positive coefficients a_{ki}, b_{ki} $(k = 1, 2, 3)$ are expressed in terms of the pneumatic-tire parameters introduced by M. V. Keldysh by means of the relations

$$a_{1i} = a_i/\alpha_i, \qquad a_{2i} = a_i\beta_i/\alpha_i, \qquad a_{3i} = a_i\gamma_i/\alpha_i + \sigma_i N_i,$$

$$b_{1i} = \sigma_i N_i/\alpha_i, \qquad b_{2i} = \sigma_i N_i \beta_i/\alpha_i, \qquad b_{3i} = \sigma_i N_i \gamma_i/\alpha_i + \rho_i N_i$$

and can be determined experimentally.

The quantities F_i, M_{xi}, $M_{\theta i}$ have a simple physical meaning: F_i represents the lateral force applied to the i-th wheel at the point K_i; M_{xi} is the moment of the forces about the longitudinal axis; $M_{\theta i}$ is the moment of the forces about an axis perpendicular to the plane of the road.

Under the second assumption, introducing a small parameter μ such that

$$\alpha_i = \mu^{-1}\alpha_i^0, \qquad \beta_i = \mu^{-1}\beta_i^0, \qquad \gamma_i = \mu^{-1}\gamma_i^0,$$

we similarly arrive at a stable surface of slow motions

$$-\alpha_i\xi_i + \beta_i\varphi_i + \gamma_i\chi_i = 0, \tag{8}$$

on which equations (3), with the variables φ_i eliminated according to (7), and the equations

$$\dot{x}_i + \dot{\xi}_i + V\theta_i + \frac{\alpha_i}{\beta_i}V\xi_i - \frac{\gamma_i}{\beta_i}V\chi_i = 0 \qquad (i = 1, 2, \dots, m). \tag{9}$$

hold. In this case the quantities F_i, M_{xi}, and $M_{\theta i}$ are determined by the formulas

$$F_i = a_i\xi_i + \sigma_i N_i \chi_i, \qquad M_{xi} = -\sigma_i N_i \xi_i - \rho_i N_i \chi_i,$$

$$M_{\theta i} = \frac{a_i b_i}{\beta_i}\xi_i - \frac{\gamma_i b_i}{\beta_i} \qquad (i = 1, 2, \dots, m). \tag{10}$$

4. Equations (5), (7) and, respectively, (9), (10) may be regarded as a generalization of the cornering hypothesis.*

Indeed, in the particular case $\theta = \chi = 0$, equations (5) and, respectively, (9) reduce to the cornering equations proposed by Rocard [(4)]. It follows from these equations that the cornering coefficient κ_1 is related to the kinematic coefficients of M. V. Keldysh by the simple dependence $\kappa_1 = \alpha/\beta$. In the particular case $\xi = \theta = 0$, $\chi = \text{const} \neq 0$, equations (5), (9) describe the phenomenon of cornering during the rolling of an inclined wheel, noted in the works of Yu. A. Echesistov [(10)] and E. A. Chudakov [(11)]. This phenomenon is characterized by its own cornering coefficient $\kappa_2 = \gamma/\beta$.

* We note that equation (5) makes it possible to establish one relation of practical importance. As is known, in order to create an axial force pressing the wheel hub against the inner bearing, the front wheels of an automobile are usually set with a small outward inclination, i.e., with "camber." But wheel inclination leads to the appearance of cornering and turning forces. As follows from equation (5), these forces can be eliminated by introducing an appropriate "toe-in" of the wheels.

Indeed, substituting into (5) the values $\dot{x} = \xi = \dot{\theta} = 0$, expressing the absence of cornering and transverse deformation of the pneumatic tire, we obtain the relation $\theta = \dfrac{\gamma}{\beta}\chi$, which determines the wheel "toe-in" angle θ as a function of the angle χ of their "camber."

It follows from what has been said that all shimmy hypotheses are obtained from the theory of M. V. Keldysh under a certain neglect of transient processes in the pneumatic tire. Thus, in the case of motion with sufficiently high speed, when the relation

$$\tau \gg \mathrm{Re}\{2[\beta_i V(1 + \sqrt{1 - 4\alpha_i/\beta_i^2})]^{-1}\},$$

is satisfied, where τ is the smallest time in the transient processes in the initial system, one may use equations (5) and (7); and in the case when the kinematic parameters $\alpha_i, \beta_i, \gamma_i$ are sufficiently large, i.e., in the case $\tau\beta_i V \gg 1$, one may use equations (9) and (10).

5. As an example, let us consider the problem of shimmy of the front wheel of a tricycle aircraft landing gear in the case of very great stiffness of the strut. For large values of the kinematic parameters α, β, γ, the equations

of motion (3) and (9) of the system under consideration are written in the form

$$A\ddot{\theta} + h\dot{\theta} - (ac + \varkappa_1 b)\xi = 0, \qquad c\dot{\theta} + \dot{\xi} + V\theta + \varkappa_1 V\xi = 0,$$

where A is the moment of inertia of the system with respect to the strut axis; c is the wheel offset; h is the coefficient of viscous friction in the damper.

Introducing the notation $\tau = AV^2(ac + \varkappa_1 b)^{-1}$, $\nu = hV(ac + \varkappa_1 b)^{-1}$, we arrive at the characteristic equation

$$\tau p^3 + (\varkappa_1 \tau + \nu)p^2 + (\varkappa_1 \nu + c)p + 1 = 0.$$

The conditions for stability of the steady motion

$$ac + \varkappa_1 b > 0, \qquad (\varkappa_1 \tau + \nu)(\varkappa_1 \nu + c) > \tau$$

differ from the conditions obtained by M. V. Keldysh [3] only in the region of self-excitation of oscillations at high frequencies.

In the case of motion with sufficiently high speed V, the characteristic equation of the system under consideration, according to (3) and (7), is written in the form

$$AV^2p^2 + [hV + c(ca_2 + b - a_1)]p + ca_2 + b = 0$$

and leads to a stability condition coinciding with the condition of M. V. Keldysh as $V \to \infty$.

Scientific Research Institute
of Applied Mathematics and Cybernetics
at Gorky State University
named after N. I. Lobachevsky

Received
23.XII 1965

REFERENCES

[1] B. A. Glukh, *Izv. NATI*, No. 1 (1935).
[2] Kontrowitz, NACA Report, No. 686 (1940).
[3] M. V. Keldysh, *Tr. TsAGI*, No. 564 (1946).
[4] Y. Rocard, *Dynamique générale des vibrations*, Paris, 1949.
[5] I. H. Greidanus, *Rapp.*, v. 1038, Nat. Luchtvaartlab., Amsterdam.
[6] Ya. M. Pevzner, *Theory of Stability of the Automobile*, 1947.
[7] E. A. Chudakov, *Shimmy of an Automobile Wheel*, 1947.

[8] I. I. Metelitsyn, *DAN*, 61, No. 3 (1948).
[9] G. V. Aronovich, *PNMM*, 13, issue 5 (1949).
[10] A. E. Chestov, Collected volume *Problems of Mechanical Engineering*, Publishing House of the USSR Academy of Sciences, 1950.
[11] E. A. Chudakov, *DAN*, 90, No. 3 (1953).
[12] K. S. Kolesnikov, *Self-Oscillations of Steerable Automobile Wheels*, 1955.
[13] G. A. Maggi, *Atti della R. Accad. Nazion. dei Lincei* ser. 5, 10, Roma (1901).
[14] A. A. Andronov, A. A. Vitt, S. E. Khaikin, *Theory of Oscillations*, Moscow, 1959.

Note: Figure translations are in progress. See original paper for figures.