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# ON THE SPECTRUM OF ERGODIC AUTOMORPHISMS

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**Abstract**

**Full Text**

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*MATHEMATICS*

**V. I. OSELEDETS**

## **ON THE SPECTRUM OF ERGODIC AUTOMORPHISMS**

*(Presented by Academician A. N. Kolmogorov on 6 X 1965)*

It is known that the spectrum of an ergodic automorphism with discrete spectrum is always simple. A. N. Kolmogorov long ago proposed an example of an ergodic automorphism with simple continuous spectrum. I. V. Girsanov<sup>(5)</sup> proved that in this example the spectrum is indeed simple. Recently A. M. Stepin<sup>(2)</sup> constructed an interesting example of a dynamical system with simple mixed spectrum. In his example the group of dyadic-rational numbers mod 1 serves as time. A. M. Stepin showed that his example refutes a number of natural conjectures concerning the spectrum of dynamical systems.

In § 2 of the present note an example of an ergodic automorphism with mixed spectrum is given. This example may be regarded as an analogue of A. M. Stepin's example. The multiplicity of the spectrum does not exceed 2 and, just as in A. M. Stepin's example, the product of two functions having continuous spectrum is a function having discrete spectrum. From the results of Ya. G. Sinai<sup>(3,4)</sup> it follows that a dynamical system of this kind does not satisfy condition (A).

In § 2 a number of examples of ergodic automorphisms with multiple continuous spectrum are also indicated. These examples have a geometric character, being "rearrangements" of intervals. The theorem of § 1 asserts that the multiplicity of the spectrum of a "rearrangement" does not exceed the number of intervals being rearranged. At the end of § 2 an example of a "rearrangement" with continuous spectrum and multiplicity not equal to 1 is given.

It has long been supposed that the group properties of spectra of automorphisms with discrete spectrum should hold, in one sense or another, also for automorphisms with continuous spectrum. The study of such properties of dynamical systems satisfying condition (A) was carried out by Ya. G. Sinai<sup>(3,4)</sup>. "Rearrangements" possess several unexpected spectral properties and, apparently, do not satisfy condition (A). It would be interesting to understand in what sense the spectra of "rearrangements" possess group properties.

§ 1. A one-to-one mod 0 transformation  $T$  of the interval  $[0, 1)$  will be called a rearrangement if

$$Tx = x + a(x), \tag{1}$$

where  $a(x) = a_k$  on the interval  $\Delta_k$ ,  $k = 1, 2, \dots, p$ ,  $\bigcup_{k=1}^p \Delta_k = [0, 1)$ ,  $\Delta_k \cap \Delta_{k'} = \emptyset$  for  $k \neq k'$ .

Excluding the uninteresting case when  $T$  has a periodic component, we shall assume that

$$\prod_{k=0}^{+\infty} T^k \xi = \varepsilon, \tag{2}$$

where  $\xi$  is the partition of  $[0, 1)$  into the intervals  $\Delta_k$ ;  $\varepsilon$  is the partition into individual points.

**Theorem.** The multiplicity function of any interval exchange  $T$  satisfying (2) does not exceed  $(p - 1)$ , the number of exchanged intervals.

The idea of the proof is as follows: it is shown that the smallest invariant subspace  $H$ , containing  $\chi_{\Delta_k}(x)$ ,  $k = 1, 2, \dots, p$ , where  $\chi_{\Delta_k}(x)$  is the indicator of the interval  $\Delta_k$ , contains a unitary subring corresponding to the partition

$$\prod_{k=0}^{n-1} T^k \xi$$

for all  $n$ , and then from (2) it follows that  $H = L^2[[0, 1]]$ .

In ergodic theory the notion of a skew product over an automorphism  $T$  is well known (see, for example, <sup>(1)</sup>). Below we consider some examples of skew products over an interval exchange  $T$ . In them the space with measure is the direct product  $[0, 1) \times Z_q$ , where  $Z_q$  is the group of roots of unity of degree  $q$  of 1; the measure is given as the direct product, and the skew product  $\tilde{T}$  has the form

$$\tilde{T}(x, y) = (Tx, g(x)y),$$

where  $g(x) \in Z_q$  for all  $x$ ; the partition of  $[0, 1)$  into the inverse images of the values of the function  $g(x)$  is a partition into a finite number of intervals. Let us note that  $\tilde{T}$  can always be realized as an interval exchange. Therefore the spectrum of  $\tilde{T}$  has finite multiplicity.

**§ 2. Example of an ergodic automorphism in which the product of any two functions with continuous spectrum is a function with discrete spectrum.** The automorphism  $T_1$  is constructed as a skew product over a rotation of the unit circle  $\Omega$ .  $T_1$  acts in  $\Omega \times Z_2$  by the formula

$$T_1(\omega, y) = (\omega + \alpha, g(\omega)y),$$

where  $\alpha$  is an irrational number,  $g(\omega) = 1$  on one half and  $g(\omega) = -1$  on the other half of the circle.  $L^2(\Omega \times Z_2)$  decomposes into the direct sum of two

invariant subspaces; the first consists of functions of the form  $f_0(\omega)$ , and the second of functions of the form  $f_1(\omega)y$ . The spectrum of  $T_1$  in the first subspace coincides with the spectrum of the rotation of  $\Omega$  through the angle  $2\pi\alpha$ . The continuity of the spectrum of  $T_1$  in the second subspace is established by means of Lemma 1.

The required property follows from the relation

$$f_1(\omega)yf_1'(\omega)y = f_1(\omega)f_1'(\omega).$$

It can be shown that the multiplicity of the spectrum of  $T_1$  does not exceed 2. It is possible that the multiplicity is equal to 1.

**Example of an interval exchange with continuous spectrum.** Let  $b = \frac{2}{7}(1 - \alpha)$ , where  $\frac{1}{2} < \alpha < 1$  and  $\alpha$  is an irrational number belonging to a set  $k$  of full measure from Lemma 2.

Divide the interval  $[0, 1)$  into four intervals:

$$\Delta_1 = [0, b), \quad \Delta_2 = [b, 1/7), \quad \Delta_3 = [1/7, 6/7), \quad \Delta_4 = [6/7, 1).$$

The automorphism  $T_2$  is defined by the formula:

$$T_2x = x + a(x),$$

where  $a(x) = 1 - b$  for  $x \in \Delta_1$ ;  $a(x) = \frac{5}{7} - b$  for  $x \in \Delta_2$ ;  $a(x) = -\frac{1}{7}$  for  $x \in \Delta_3$ ;  $a(x) = -b$  for  $x \in \Delta_4$ .

The continuity of the spectrum of  $T_2$  is established by means of Lemma 2. It can be shown that the multiplicity of the spectrum of  $T_2$  does not exceed 2. It is possible that the multiplicity is equal to 1.

**Example of an interval exchange with a multiplicity function not equal to 1.** First we carry out an auxiliary construction. The automor-

the automorphism  $T_3$  acts in  $[0, 1) \times Z_2$  according to the formula

$$T_3(x, z) = (T_2x, n(x)z),$$

where  $n(x) = -1$  for  $x \in [0, 5/7)$ ;  $n(x) = 1$  for  $x \in [5/7, 1)$ .

The continuity of the spectrum of  $T_3$  is established with the aid of Lemma 1.

The example itself is constructed as follows. The desired automorphism  $T_4$  is defined by the formula

$$T_4(x, z, y) = (T_3(x, z), \varphi(x, z)y),$$

where  $y \in Z_3$ ,  $\varphi(x, z) = \cos 2\pi/3 + iz \sin 2\pi/3$ .

The continuity of the spectrum of  $T_4$  is established with the aid of Lemma 3. The functions  $f_1(x, z, y) = y$  and  $f_2(x, z, y) = y^2$  generate orthogonal cyclic subspaces. The spectra of these functions coincide, since the Fourier coefficients of these functions coincide. The coincidence of the Fourier coefficients follows from the equality

$$\varphi(x, -z) = \varphi^2(x, z).$$

§ 3. The results of § 2 are based on the lemmas formulated in this section.

**Lemma 1.** *Let  $T$  be an automorphism of the space  $\Omega$ , and let  $\sigma$  be an automorphism commuting with  $T$ . If  $g(\omega)$  is an eigenfunction for  $\sigma$  with an eigenvalue not belonging to the discrete component of the spectrum of  $T$ , then the equation*

$$f(T\omega) = g(\omega)f(\omega),$$

where  $|f(\omega)| = 1$ , has no measurable solutions.

**Lemma 2.** *Let  $F(\omega) = 1$  on one half and  $F(\omega) = -1$  on the other half of the unit circle  $\Omega$ . If  $3\lambda$  is not an integer, then the equation*

$$f(\omega + \alpha) = e^{2\pi i \lambda F(\omega)} f(\omega),$$

where  $|f(\omega)| = 1$ ,  $\alpha \in K$ —a set of full measure\*, has no measurable solutions.

**Lemma 3.** *Let*

$$B(\omega) = \begin{pmatrix} \cos 2\pi/p & \sin 2\pi/p \\ \sin 2\pi/p & -\cos 2\pi/p \end{pmatrix}$$

on one half of the circle and

$$B(\omega) = \begin{pmatrix} \cos 4\pi/p & \sin 4\pi/p \\ \sin 4\pi/p & -\cos 4\pi/p \end{pmatrix}$$

on the other half of the circle  $\Omega$ , where the integer  $p$  is not less than 3. Then the equation

$$j(\omega + \alpha) = B(\omega)j(\omega),$$

where  $j(\omega)$  is a two-dimensional vector-function,  $|j(\omega)| = 1$ , for almost all  $\alpha$  has no measurable solutions.

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\* We give the definition of the set  $K$ : let  $p_n/q_n$  be the  $n$ -th convergent of the continued fraction of  $\alpha$ ; then  $\alpha \in K$  if among the  $m_n = q_{n-1} + q_n$  there occur infinitely many odd numbers. It is not difficult to show that  $K$  has full measure.

*Note: Figure translations are in progress. See original paper for figures.*

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