TOPOLOGICAL EQUIVALENCE

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Abstract

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MATHEMATICS

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TOPOLOGICAL EQUIVALENCE

OF ALL SEPARABLE BANACH SPACES

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In this note the following is proved.

Theorem. *All separable Banach spaces are homeomorphic.*

Thus an affirmative answer is obtained to the question posed by M. Fréchet (^{1}) in 1928 and S. Banach (^{2}) in 1932.

C. Bessaga and A. Pełczyński (^{3}) showed that if a separable Banach space E contains a subspace X homeomorphic to l_2, then it itself is also homeomorphic to l_2. It is well known that every infinite-dimensional Banach space contains an infinite-dimensional subspace with a basis. Hence, by the Bessaga-Pełczyński theorem, it is enough to establish a homeomorphism of all Banach spaces with a basis.

Proposition 1. *Let X be a Banach space with basis $\{e_k\}_1^\infty$; denote the conjugate basis system by $\{f_k\}_1^\infty$ $(f_k \in X^*)$. Suppose, further, that the norm of the space is subject to the following condition: if*

$$\|x_\nu\| = \|x\| = 1 \quad (\nu = 1, 2, \ldots); \qquad \lim_{\nu\to\infty} f_n(x_\nu) = f_n(x) \quad (n = 1, 2, \ldots),$$

then

$$\lim_{\nu\to\infty} \|x_\nu - x\| = 0.$$

Assume also that on the unit ball U of the space X there is defined a functional $F(x)$ having the following properties:

1. *The functional $F(x)$ is continuous.*
2. $F(x) > 0$ for $\|x\| < 1$; $F(x) = 0$ for $\|x\| = 1$; $F(0) = 1$.
3. *If*

$$\lim_{n\to\infty} F(S_n) = 0 \qquad \left(S_n = \sum_{k=1}^{n} a_k e_k\right), \tag{1}$$

then the series $\sum_{k=1}^{\infty} a_k e_k$ *converges.*

4. *For fixed* n *and* a_k $(k < n)$, *the function*

$$\psi(a) = F(S_{n-1} + a e_n)$$

is strictly increasing for $a < 0$ *and strictly decreasing for* $a > 0$.

Then the space X *is homeomorphic to the space* l_2.

We shall precede the proof of Proposition 1 by two lemmas. To each normalized element

$$x = \sum_{k=1}^{\infty} f_k(x) e_k$$

we assign a pair of numerical sequences:

$$\gamma_0(x) = 1; \qquad \gamma_n(x) = F(S_{nx}) \quad \left(S_{nx} = \sum_{k=1}^{n} f_k(x) e_k,\ n = 1, 2, \ldots\right),$$

$$\theta_n(x) = \operatorname{sign} f_n(x) \quad (n = 1, 2, \ldots).$$

We note that

$$1 \geq \gamma_1(x) \geq \gamma_2(x) \geq \cdots, \qquad \lim_{n\to\infty} \gamma_n(x) = 0.$$

Lemma 1. A normalized sequence x_ν converges to an element x if and only if

$$\lim_{\nu\to\infty} \theta_n(x_\nu)\,[\gamma_{n-1}(x_\nu) - \gamma_n(x_\nu)] = \theta_n(x)\,[\gamma_{n-1}(x) - \gamma_n(x)] \qquad (n = 1, 2, \ldots).$$

Lemma 2. Whatever the pair of numerical sequences $\{\gamma_k\}_1^\infty$ and $\{\theta_k\}_1^\infty$, subject to the conditions

$$1 \geq \gamma_1 \geq \gamma_2 \geq \cdots, \qquad \lim_{n\to\infty} \gamma_n = 0,$$

$$\theta_n = \pm 1, \quad \text{if } \gamma_n \neq \gamma_{n-1}; \qquad \theta_n = 0, \quad \text{if } \gamma_n = \gamma_{n-1},$$

there exists a unique normalized element x such that

$$\gamma_n(x) = \gamma_n, \qquad \theta_n(x) = \theta_n \qquad (n = 1, 2, ...).$$

Proof of Proposition 1. Let X and Y be spaces satisfying all the requirements of Proposition 1. Suppose that in each of them the functionals $\gamma_n(x), \theta_n(x)$ and, respectively, $\gamma_n(y), \theta_n(y)$ have been introduced. To each normalized element $x \in X$ we assign that normalized element $y = Tx \in Y$ for which

$$\theta_n(y)\gamma_n(y) = \theta_n(x)\gamma_n(x) \qquad (n = 1, 2, ...).$$

By Lemma 2 this correspondence is one-to-one. By Lemma 1 it is bicontinuous. The homeomorphism thus obtained extends to the whole space X:

$$Tx = \|x\| T(x/\|x\|), \qquad T(\theta) = \theta.$$

It remains to show that in the space l_2 there exists a functional $F(x)$ with the required properties. It turns out that in this case it is sufficient to put $F(x) = 1 - \|x\|$ $(x \in l_2)$.

We now show that every Banach space with a basis can, by means of an equivalent renorming, be made to satisfy the conditions of Proposition 1.

Proposition 2. In every separable Banach space X with basis $\{e_k\}_1^\infty$ one can introduce a norm ($\| \ \|$), equivalent to the original one ($\| \ \|_0$), satisfying the following requirements:

a) The space $(X, \| \ \|)$ is locally uniformly convex (UR_L in the notation of M. Day, (^{4}), p. 188);

b) For any normalized elements x_ν and x, the condition

$$\lim_{\nu \to \infty} f_n(x_\nu) = f_n(x) \qquad (n = 1, 2, ...)$$

implies strong convergence:

$$\lim_{\nu \to \infty} \|x_\nu - x\| = 0.$$

c) With respect to the new norm the basis $\{e_k\}$ is orthogonal, i.e.

$$\left\| \sum_{k=1}^{n-1} a_{ke} k \right\| < \left\| \sum_{k=1}^{n} a_{ke} k \right\|, \qquad \text{if } a_n \neq 0 \quad (n = 1, 2, ...).$$

A norm satisfying the indicated requirements was constructed in (^{5,6}) (meaning the norm defined by formula (3) of the note (^{6})).

Proposition 3. In a Banach space whose norm satisfies the conditions a), b), c) of Proposition 2, there exists a functional $F(x)$ possessing properties 1-4 formulated in Proposition 1.

Here we shall need two more lemmas. Consider the functional

$$\varepsilon(x,\delta)=\frac{1}{2}\sup_{z\in G(x,\delta)}\|x-z\| \qquad (\|x\|=1,\ 0\le\delta\le 1),$$

where

$$G(x,\delta)=\{z:\|z\|\le 1;\ \min_{0\le\lambda\le 1}\|\lambda x+(1-\lambda)z\|\ge 1-\delta\}.$$

If the space is locally uniformly convex, then

$$\lim_{\delta\to 0}\varepsilon(x,\delta)=0. \tag{2}$$

Lemma 3. For any Banach space the functional $\varepsilon(x,\delta)$ satisfies the inequalities:

$$\delta\leqslant\varepsilon(x,\delta)\leqslant 1, \qquad \varepsilon(x,\delta)\leqslant\tfrac{1}{2}\|x-y\|+\varepsilon(y,\delta+\|x-y\|)$$

$$\varepsilon(x,\delta+h)-\varepsilon(x,\delta)\leqslant 3h/\delta \qquad (0\leqslant\delta\leqslant\delta+h\leqslant 1), \tag{3}$$

whence, in particular, it follows that $\varepsilon(x,\delta)$ is uniformly continuous on the set $S\times[\delta_0,1]$ for every $\delta_0>0$ (S is the unit sphere of the space). If the space is locally uniformly convex, then $\varepsilon(x,\delta)$ is continuous on $S\times[0,1]$.

Let us consider the functional

$$\Phi(x)=\varepsilon\left(\frac{x}{\|x\|},\ 1-\|x\|\right) \qquad (0<\|x\|\leqslant 1);$$

for $x=\theta$ put $\Phi(\theta)=1$.

Lemma 4. The functional $\Phi(x)$ is continuous for $\|x\|\leqslant 1$ and uniformly continuous for $\|x\|\leqslant 1-\delta_0$ for every $\delta_0>0$.

For each element x ($\|x\|\leqslant 1$) define the set

$$L(x)=\bigcup_n L_n(x),$$

where $L_n(x)$ is the segment joining $S_{n-1}x$ and S_{nx} ($n=1,2,\ldots$); $S_0x=\theta$. Finally, define the functional

$$F(x)=\left(1-\tfrac{1}{2}\|x\|\right)\inf_{z\in L(x)}\Phi(z) \qquad (0\leqslant\|x\|\leqslant 1). \tag{4}$$

Proof of Proposition 3. We have to verify that the functional $F(x)$ has properties 1-4 of Proposition 1. Properties 1 and 2 follow almost immediately from the definition of $F(x)$ and Lemma 4. Let us prove property 3. Suppose (1) is fulfilled. For each S_n consider a point $\sigma_n \in L(S_n)$ at which the lower bound in expression (4) is attained:

$$\sigma_n = S_{m-1} + \alpha e_m \qquad (m = m(n),\ 0 < |\alpha| \leqslant |a_m|,\ \alpha a_m > 0). \tag{5}$$

According to (2), (3), and (5), the diameter of the set $G(\sigma_n/\|\sigma_n\|;\, 1 - \|\sigma_n\|)$ tends to zero as n increases. Consider also the decreasing sequence of closed sets

$$Q(\sigma_n) = \{z:\ \|z\| \leqslant 1;\quad f_k(z) = a_k\ (k = 1, 2, \dots, m)\} \qquad (n = 1, 2, \dots).$$

Using the orthogonality of the basis $\{e_k\}$, one can establish that

$$G(\sigma_n/\|\sigma_n\|;\, 1 - \|\sigma_n\|) \supset Q(\sigma_n).$$

Thus the diameter of the set $Q(\sigma_n)$ tends to zero as n increases. The unique element x lying in the intersection of all $Q(\sigma_n)$ will also be the limit of the sequence

$$S_n: \quad x = \sum_{k=1}^{\infty} a_k e_k.$$

Finally, let us prove property 4. Consider the function

$$\psi(\alpha) = F(S_{n-1} + \alpha e_n).$$

Let $|\alpha_1| < |\alpha_2|$, $\alpha_1\alpha_2 \geqslant 0$. Then

$$\|S_{n-1} + \alpha_1 e_n\| < \|S_{n-1} + \alpha_2 e_n\| \tag{6}$$

by virtue of the orthogonality of the basis, and

$$L(S_{n-1} + \alpha_1 e_n) \subset L(S_{n-1} + \alpha_2 e_n) \tag{7}$$

by the definition of the set $L(x)$. From (4), (6), and (7) we obtain that $\psi(\alpha_1) > \psi(\alpha_2)$.

The theorem formulated at the beginning of the note is a direct consequence of Propositions 1-3 and the theorem of Bessaga-Pełczyński.

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Note: Figure translations are in progress. See original paper for figures.

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