ON THE FORMULATION OF BOUNDARY-VALUE PROBLEMS FOR DEGENERATE ELLIPTIC EQUATIONS

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Abstract

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ON THE FORMULATION OF BOUNDARY-VALUE PROBLEMS FOR DEGENERATE ELLIPTIC EQUATIONS

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Let, in the n-dimensional space R^n, there be given an elliptic, possibly degenerate, operator

$$Lu(x) = \sum_1^n a_{ij}(x)u_{x_i x_j} + \sum_1^n b_i(x)u_{x_i}.$$

The coefficients of the operator are assumed to be continuously differentiable. The present note is devoted to the study of uniqueness conditions for the solution of boundary-value problems for the operator L. In particular, this circle of questions includes the problem of uniqueness classes for degenerate parabolic equations, theorems of the Phragmén-Lindelöf principle type for degenerate elliptic equations, and uniqueness conditions in the case of a bounded domain. The note also studies the question of stabilizing the solution of a mixed problem for parabolic equations. By a solution of a boundary-value problem we mean a generalized solution in the probabilistic sense (^{1}). In (^{2}) some results are given concerning the coincidence of a probabilistic solution and a generalized solution in the sense of an integral identity. The smoothness of generalized solutions is investigated in (^{3,7}).

1°. Let D be a bounded domain in R^n with smooth boundary Γ, and let $\{n_i(x)\}$ be the direction cosines of the outward normal. In the papers ($^{1-7}$) the problem

$$Lu(x) = 0 \quad \text{for } x \in D, \qquad \lim_{x \to y \in \tilde{\Gamma}} u(x) = \psi(y), \tag{1}$$

was studied, where $\tilde{\Gamma}$ is the regular part of the boundary Γ (see (^{1,8})), and $\psi(y)$ is a continuous function on $\tilde{\Gamma}$. For regularity one can give simple sufficient conditions in terms of the coefficients. For example, for regularity of a point $x \in \Gamma$ it is sufficient that

$$x \in \left\{x : \sum a_{ij}(x)n_i(x)n_j(x) \neq 0\right\} \cup \left\{x : \sum_i \left[b_i(x) - \sum_j \frac{\partial a_{ij}}{\partial x_j}\right] n_i(x) > 0\right\}.$$

Some criteria of regularity can be found in (^{8}). In (^{1}) a generalized solution of problem (1) is constructed and conditions ensuring uniqueness are given. In particular, for uniqueness it is sufficient that the following be satisfied.

Condition A. The domain D can be covered by two sets D_1 and D_2 such that

$$\inf_{x \in D_1} a_{ii}(x) > 0, \qquad \inf_{x \in D_2} b_i(x) > 0$$

or

$$\sup_{x \in D_2} b_i(x) < 0$$

for some $i = 1, \dots, n$.

In this paragraph we shall give a certain natural refinement of problem (1). This refined problem will always have a unique solution. Put

$$k_i(x) = b_i(x) - \sum_j \frac{\partial a_{ij}(x)}{\partial x_j}.$$

We say that a smooth curve $\gamma = \{x_1(t), \dots, x_n(t),\ x_n(t),\ t \in [0,1]\}$, joining the points a and b, is **passable** from a to b if

$$\gamma = \gamma_1 \cup \gamma_2, \quad \text{where} \quad \gamma_1 \subset \left\{ x : \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial x_i}{\partial t} \frac{\partial x_j}{\partial t} > 0 \right\};$$

γ_2 is a part of a characteristic of the equation

$$\sum l_i(x) \frac{\partial V}{\partial x_i} = 0,$$

and the motion along γ_2 takes place in the direction of b, and

$$\inf_{\gamma_2} \sum_1^n b_i^2(x) > 0.$$

We shall call a surface S **impassable** from a to b if S separates a from b in the domain D,

$$\sum a_{ij}(x) h_i(x) h_j(x) \Big|_{x \in S} = 0$$

and

$$\sum l_i(x) h_i(x) \Big|_{x \in S} \geq 0,$$

where $\{h_i(x)\}$ are the direction cosines of the normal to S directed toward a.

Assume that in $D \cup (\tilde{\Gamma} \setminus \Gamma)$ there exists a set Π with the following properties:

1. For every $x \in D \cup (\Gamma \setminus \Gamma)$ there is an $a(x) \in \Pi$, limiting for points a_i, which are joined with x by curves passable from x to a_i. Let

   $$S_a = \{x :\ a(x) = a\} \cap \{x : \rho(x, a) < \varepsilon\}$$

   for some $\varepsilon > 0$ be a smooth manifold.

2. The set $\Pi = \Pi_1 + \Pi_2$. Each point $a \in \Pi_1$ has a neighborhood $U(a)$ such that the points of the set $U(a) \cap S_a$ are separable from points in $\Pi \setminus a$, and the points in $U(a) \setminus S_a$ are separable from a.
3. The closed set Π_2 is the sum of nonintersecting manifolds π_α (the manifolds may contain their boundary). Each π_α is an attracting set* in the sense of (8), if it is regarded as part of the boundary of the set $D \setminus \pi_\alpha$. By Π_2^1 we shall denote the union of those π_α which consist of regular points for the problem in $D \setminus \pi_\alpha$.

Theorem 1. *If, for the operator L in the domain D, there exists a set Π with the properties indicated above, then there exists a unique solution of the following problem:*

Problem $\mathcal{L}$. $Lu(x) = 0$ for $x \in D$;

$$\lim_{x \to y \in \Gamma} u(x) = \psi(y), \qquad \lim_{\substack{x \to y \in \Pi \\ x \in S_y}} u(x) = \varphi(y),$$

where $\varphi(y)$ is a bounded function, continuous on Π_2^1 and equal to a constant on each $\pi_\alpha \subset \Pi_2 \setminus \Pi_2^1$.

Theorem 2. *Suppose that the conditions of Theorem 1 are satisfied and $\Pi = \Pi_1$. Let the operator L be nondegenerate on Γ and, for $x \in \Pi$, be nondegenerate if it is considered on the manifold S_x (i.e., after discarding all terms containing derivatives in directions normal to S_x). Then every bounded solution of problem (I) is a solution of problem $\mathcal{L}$ with some function $\varphi(x)$.*

Remark. From the theorems formulated it follows that the set $\tilde{\Gamma} \cup \Pi$ is the Martin boundary for the operator L in the domain D (if the conditions of Theorem 2 are satisfied). Theorem 1 may also be regarded as a description of the proper space of the operator L with proper value zero.

The following theorem shows that the solution of the mixed problem

$$\partial v(t,x)/\partial t = Lv, \qquad v(0,x) = f(x), \qquad v(t,x)|_{x \in \Gamma} = \psi(x) \tag{II}$$

in the domain $D \times [0, \infty)$ stabilizes to the solution of a certain problem $\mathcal{L}$.

Theorem 3. *Suppose that the conditions of Theorem 1 are satisfied and the set Π can be represented as the sum of three sets $\Pi = \Pi_I + \Pi_{II} + \Pi_{III}$ such that:*

1. If $x \in \Pi_I$, then the operator L does not degenerate at the point x, if it is considered on S_x.

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* It is most convenient to define the notion of an attracting set in terms of the Markov process corresponding to the operator L. In order that a set E be attracting, it is enough that there exist a function $v(x)$, equal to zero on E and positive in a neighborhood E, such that $Lv(x) \leq 0$ in a neighborhood $\dot{E}$. Sufficient conditions can be formulated in terms of the coefficients.

2. $\Pi_{\mathrm{II}} = \Pi_2^1$.
3. Each point of Π_{III} belongs to some closed integral curve of the field $\{b_i(x)\}$, and along this curve $a_{ij}(y) = 0,\ f(y) = \mathrm{const}$.

Then the solution of problem (II), as $t \to \infty$, has a limit $u(x)$ for every $x \in D$. The function $u(x)$ is a solution of problem I, and $\varphi(x)$ is determined as follows:

$$\varphi(x) = f(x), \quad \text{if } x \in \Pi_{\mathrm{II}} \cup \Pi_{\mathrm{III}}; \qquad \varphi(x) = \int_{A_x} f(y)\, \mu_x(dy),$$

if $x \in \Pi_I$, where $\mu_x(\cdot)$ is a measure on the set A_x of points that are connected with $x \in \Pi$ by an admissible curve. The measure $\mu_x(\cdot)$ is a solution (possibly generalized) of the equation $L^* g = 0$ on A_x with the conditions of nonnegativity and $\mu_x(A_x) = 1$.

Let us note that the theorems of this paragraph are essentially connected with the classification of states of Markov chains and with ergodic theorems for chains.

2°. In this paragraph we consider uniqueness classes of solutions of the Cauchy problem for a degenerate parabolic equation. From (1) there follows uniqueness in the class of bounded functions if the coefficients of the operator are bounded in the whole space.

Theorem 4. In order that the solution of the Cauchy problem for the equation

$$\frac{\partial u}{\partial t} = Lu(t, x)$$

be unique in the class of all continuous functions without restrictions on growth as $|x_i| \to \infty$, it is sufficient that at least one of the following conditions be satisfied.

1. There exists a sequence of nested open sets $G_1, \dots, G_n, \dots$ with smooth boundaries $\Gamma_1, \dots, \Gamma_n, \dots$, whose union gives the whole space, such that the surface Γ_i is not reachable from points $x \in G_i$ to points of the domain $R^n \setminus (G_i \cup \Gamma_i)$.
2. There exists an expanding sequence of spherical layers
$$S_i = \{x : \ \alpha_i < r(x) < \beta_i\}$$
(the layers do not intersect and S_i separates S_{i-1} from S_{i+1}) such that $a_r(x) = 0$ for
$$r(x) \in \bigcup_i (\alpha_i, \beta_i),$$
$$\sum_{i=N}^{\infty} \int_{\alpha_i}^{\beta_i} \frac{dr}{\max_\varphi b_r(r, \varphi)} = \infty$$
for any $N > 0$. Here $r(x), \varphi(x)$ are the spherical coordinates of the point x; $a_r(x), b_r(x)$ are the coefficients of $\partial^2/\partial r^2$ and $\partial/\partial r$ in the operator L, written in spherical coordinates.

Remark 1. Condition 2 can be weakened; however, if there exists a curve going to infinity along which the diffusion does not degenerate, then there will no longer be uniqueness for arbitrary growth.

Remark 2. If the conditions of Theorem 4 are satisfied, then the solution of the Cauchy problem has a finite domain of dependence.

Theorem 5. If the coefficients of the operator L are bounded in the whole space, then there exists a unique solution of the Cauchy problem for the equation

$$\frac{\partial u}{\partial t} = Lu(t, x)$$

in the class of functions growing, as $|x| \to \infty$, more slowly than $e^{c|x|^2}$ for some $c > 0$.

Remark 1. If there exists a smooth curve γ going to infinity such that

$$\lim_{r\to\infty} \frac{\gamma(r)}{r} = a > 0,$$

where $\gamma(r)$ is the length of the curve γ up to the first attainment of the sphere of radius r with center at zero, and along which the diffusion is uniformly nondegenerate, then the uniqueness class indicated in Theorem 5 is close to optimal.

If one allows growth of the coefficients at infinity, then the uniqueness class may narrow to the class of functions having a prescribed limit at infinity; moreover, in different directions, generally speaking, different limits may be prescribed. Some results concerning uniqueness classes in the case when the equation degenerates only at infinity were obtained in (9).

3°. In this paragraph we formulate several theorems of the Phragmén–Lindelöf principle type. Consider in the space R^{n+1} the cylinder $T = D \times [0, \infty)$. We assume that the x_{n+1}-axis is directed along the generator of the cylinder, D is a bounded domain in the subspace $R^n = \{x_{n+1} = 0\}$, and Γ is the boundary of the domain D. Consider in the cylinder T the Dirichlet problem:

$$Lu(x) = \sum_1^{n+1} a_{ij}(x) u_{x_i x_j} + \sum_1^{n+1} b_i(x) u_{x_i} = f(x), \qquad u(x)\big|_{\Gamma\times[0,\infty)\cup D} = 0. \quad \text{(III)}$$

Theorem 6. Suppose that for the operator L condition A is satisfied for $i \leq n$, and suppose, moreover, that one of the following conditions is satisfied.

1. There exists a sequence α_k, $\lim_{k\to\infty} \alpha_k = \infty$, such that

$$a_{n+1,n+1}(x_1, \dots, x_n, \alpha_k) = 0, \quad b_{n+1}(x_1, \dots, x_n, \alpha_k) \leq 0.$$

2. There exists a sequence of nonintersecting intervals $[\alpha_k, \beta_k]$, $\lim_{k\to\infty} \alpha_k = \infty$, such that $a_{n+1,n+1}(x) = 0$ for $x_{n+1} \in \bigcup_k [\alpha_k, \beta_k]$,

$$\sum_N^{\infty} \int_{\alpha_k}^{\beta_k} \frac{ds}{\max\limits_{(x_1,\dots,x_n)\in D} b_{n+1}(x_1, \dots, x_n, s)} = \infty \quad \text{for any } N.$$

Then the solution of problem (III) is unique in the class of all continuous functions without restrictions on growth as $x_{n+1} \to \infty$.

Theorem 7. Suppose that condition A is satisfied for $i = 1, \dots, n$, and suppose that the coefficients of the operator L are bounded in T. Then the solution of problem (III) is unique in the class of functions growing as $x_{n+1} \to \infty$ more slowly than $e^{\lambda x_{n+1}}$, where the positive number λ depends on the coefficients of the equation and on the diameter of the domain D.

Theorem 8. Suppose that the coefficients of the operator L are bounded and that for some $i \leq n$, for $x \in T$, the relations $a_{ii}(x) \equiv 0$, $b_i(x) > \alpha > 0$ are satisfied. Then the solution of problem (III) is unique in the class of functions growing in x_{n+1} more slowly than $e^{c x_{n+1}{}^2}$ for some $c > 0$.

If condition A is not satisfied and, for every x_{n+1}, in the domain D there exists the set Π (see Theorem 1), then for uniqueness one must prescribe the limit $u(x)$ as $x_{n+1} \to \infty$ for some points of the domain D. The uniqueness class may narrow to the set of functions having a prescribed limit as $x_{n+1} \to \infty$ (its own for different $(x_1, \dots, x_n) \in D$), if the coefficients are allowed to grow sufficiently rapidly.

In conclusion we note that the proof of all the theorems formulated here is carried out by considering trajectories of the corresponding Markov processes.

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Note: Figure translations are in progress. See original paper for figures.

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