A VOTING FUNCTION FOR CONTINUOUS QUANTITIES

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Abstract

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A VOTING FUNCTION FOR CONTINUOUS QUANTITIES

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In binary logic the voting function is often used

$$y = \operatorname{maj}(x_1, x_2, \dots, x_n), \tag{1}$$

which takes the value 0 if the majority of the independent binary variables $(x_1, x_2, \dots, x_n)$ is zero, and the value 1 if the majority of these variables is one. The voting function (1) is also called the majority function. This function can be written either in conjunctive normal form or in disjunctive normal form.

Let there be an odd number $(n = 2k - 1)$ of independent binary variables $(x_1, x_2, \dots, x_{2k-1})$; then the voting function in conjunctive normal form has the form

$$y = \operatorname{maj}(x_1, x_2, \dots, x_{2k-1}) = u_1 u_2 u_3 \dots u_{C_{2k-1}^k}, \tag{2}$$

where $u_1, u_2, \dots, u_{C_{2k-1}^k}$ are all possible logical sums composed of $2k-1$ variables taken k at a time (C_{2k-1}^k is the number of combinations of $2k-1$ taken k):

$$\begin{aligned} u_1 &= x_1 + x_2 + \dots + x_{k-1} + x_k, \\ u_2 &= x_1 + x_2 + \dots + x_{k-1} + x_{k+1}, \\ &\dots\dots\dots\dots\dots\dots\dots \\ u_{C_{2k-1}^k} &= x_k + x_{k+1} + \dots + x_{2k-2} + x_{2k-1}. \end{aligned} \tag{3}$$

In disjunctive normal form

$$y = \operatorname{maj}(x_1, x_2, \dots, x_{2k-1}) = v_1 + v_2 + v_3 + \dots + v_{C_{2k-1}^k}, \tag{4}$$

where $v_1, v_2, \dots, v_{C_{2k-1}^k}$ are all possible logical products composed of $2k-1$ variables taken k at a time:

$$\begin{aligned} v_1 &= x_1 x_2 \dots x_{k-1} x_k, \\ v_2 &= x_1 x_2 \dots x_{k-1} x_{k+1}, \\ &\dots\dots\dots\dots\dots \\ v_{C_{2k-1}^k} &= x_{kx_{k+1}} \dots x_{2k-2} x_{2k-1}. \end{aligned} \tag{5}$$

One of the most important applications of the voting function is its use for identifying the true value of a signal in digital systems with parallel redundancy (^{1}). In this case the voting function (1) is often called a restoring function, and the element that implements this function is called a restoring unit.

It appears possible to define a certain voting function (or restoring function) also for continuous signals, on the basis of which restoring units can be created for analog systems with parallel redundancy. We shall use the known relationship between logical operations on binary and continuous quantities (^{2}) to find this function.

The disjunction function for binary variables corresponds to the function selecting the maximum of continuous variables:

$$x_1 + x_2 + x_3 + \dots \sim \max(x_1, x_2, x_3, \dots), \tag{6}$$

and the conjunction function corresponds to selecting the minimum of continuous variables

$$x_1 x_2 x_3 \dots \sim \min(x_1, x_2, x_3, \dots). \tag{7}$$

It is obvious that if the variables $x_1, x_2, x_3, \dots$ on the right-hand side of (6) and (7) take only the values 0 and 1, then the correspondence sign in these formulas may be replaced by an equality sign.

Using relations (6) and (7), equations (2)—(5) can be rewritten for continuous signals in the following form:

$$y = \operatorname{maj}(x_1, x_2, \dots, x_{2k-1}) = \min(u_1, u_2, u_3, \dots, u_{C_{2k-1}^k}), \tag{8}$$

where

$$\begin{aligned} u_1 &= \max(x_1, x_2, \dots, x_{k-1}, x_k), \\ u_2 &= \max(x_1, x_2, \dots, x_{k-1}, x_{k+1}), \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ u_{C_{2k-1}^k} &= \max(x_k, x_{k+1}, \dots, x_{2k-2}, x_{2k-1}), \end{aligned} \tag{9}$$

and

$$y = \operatorname{maj}(x_1, x_2, \dots, x_{2k-1}) = \max(v_1, v_2, v_3, \dots, v_{C_{2k-1}^k}), \tag{10}$$

where

$$\begin{aligned} v_1 &= \min(x_1, x_2, \dots, x_{k-1}, x_k), \\ v_2 &= \min(x_1, x_2, \dots, x_{k-1}, x_{k+1}), \\ &\dots\dots\dots\dots\dots\dots\dots \\ v_{C_{2k-1}^k} &= \min(x_k, x_{k+1}, \dots, x_{2k-2}, x_{2k-1}). \end{aligned} \tag{11}$$

It is interesting to note that expression (10) for the voting function of continuous signals was obtained by V. I. Varshavskii on the basis of considering many-valued majority logic (3). In practice, however, it is often more convenient to use expression (8).

Let us now clarify some properties of functions (8) and (10).

Let k or more independent variables have the same value, equal to x, while the remaining variables have arbitrary values. Then from (9) we find that all $u_j \geq x$, $j = 1, 2, \dots, C_{2k-1}^k$, and at the same time at least one of the values $u_j = x$. Therefore, from (8) we have $y = x$.

Under the same conditions, from (11) we have that all $v_j \leq x$, $j = 1, 2, \dots, C_{2k-1}^k$, and at the same time at least one of the values $v_j = x$. Therefore, from (10) as well we have $y = x$.

Thus, if the majority of independent continuous variables have the same magnitude, then the voting function defined by formula (8) or (10) takes this same magnitude, independently of the values of the other variables.

Theorem. Let the voting function y depend on an odd number of continuous variables, n of which take the value x. Then $y = x$, if

$$n \geq 1 + |n_{\mathrm{m}} - n_{\mathrm{b}}|,$$

where n_{m} is the number of variables whose magnitude is less than x, and n_{b} is the number of variables whose magnitude is greater than x.

Proof. According to (8), $y = x$, if: a) all $u_j \geq x$, $j = 1, 2, \dots, C_{2k-1}^k$, and b) for at least one value of j the equality is valid—

at $u_j = x$. From (9) we find that the first condition is satisfied when $n + n_{\mathrm{b}} \geq k$, and the second when $n + n_{\mathrm{m}} \geq k$.

Taking into account that $n + n_{\mathrm{m}} + n_{\mathrm{b}} = 2k - 1$, these two inequalities can be rewritten in the form $n \geq 1 + n_{\mathrm{m}} - n_{\mathrm{b}}$, $n \geq 1 + n_{\mathrm{b}} - n_{\mathrm{m}}$. Combining both expressions for n, we obtain

$$n \geq 1 + |n_{\mathrm{b}} - n_{\mathrm{m}}|, \tag{12}$$

which was what had to be proved. An analogous proof can be obtained starting from equations (10) and (11).

Corollary. Renumber the variables in the order of increasing (or decreasing) values assumed by them:

$$x_1 \leq x_2 \leq \ldots \leq x_k \leq \ldots \leq x_{2k-2} \leq x_{2k-1}.$$

Then it follows from the theorem proved that the voting function (y) of an odd number of analog signals is equal to the signal that is median in magnitude; in other words, $y = x_k$.

This property of the voting function as applied to three variables was noted by H. M. Paynter (see the discussion to (2), pp. 281-284).

From the theorem presented and its corollary it is clear that a restoring element that performs the voting function can ensure the normal operation of analog systems with parallel redundancy even when the majority of the parallel channels fail, provided that condition (12) is satisfied.

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Note: Figure translations are in progress. See original paper for figures.