ON THE RATE OF CONVERGENCE

MATHEMATICS

1966

SovietRxiv

Abstract

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UDC 519.21

MATHEMATICS

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ON THE RATE OF CONVERGENCE TO STABLE DISTRIBUTION LAWS

(Presented by Academician Yu. V. Linnik, 14 VI 1965)

Let the function $F(x)$ belong to the domain of normal attraction of the stable law $G_\alpha(x)$. This, as is known, means that, with a suitable choice of centering coefficients A_n and normalizing coefficients of the form

$$B_n = an^{1/\alpha} \tag{1}$$

the relation

$$F_n(x) = F^{*n}(B_{nx} + A_n) = G_\alpha(x) + o(1).$$

holds.

The stable law $G_\alpha(x)$, as usual, is specified by the canonical form of the logarithm of the characteristic function as follows:

$$\log \varphi_\alpha(t) = i\gamma t - C|t|^\alpha \{1 + i\beta \operatorname{sgn} t \, \omega(t, \alpha)\}, \tag{2}$$

where

$$\omega(t, \alpha) = \begin{cases} \tan \dfrac{\pi}{2}\alpha, & (\alpha \neq 1), \\ \dfrac{2}{\pi} \log |t|, & (\alpha = 1), \end{cases} \tag{3}$$

and the constants α, C, β satisfy the inequalities $0 < \alpha \leq 2$, $C > 0$, $-1 \leq \beta \leq 1$. The constant γ, which may be any real number, is taken in this note to be equal to zero, which, of course, does not restrict generality in the study of convergence to the laws $G_\alpha(x)$.

The rate of convergence of the functions $F_n(x)$ to the stable law $G_\alpha(x)$ under conditions of normal attraction is studied. This problem was considered by G.

Kramer ([1]) under rather particular assumptions concerning the function $F(x)$. The form chosen here for estimating the remainder term in relation (2) makes it possible, in the cases $0 < \alpha < 1$ and $1 < \alpha < 2$, to obtain universal estimates of the rate of convergence without any additional assumptions concerning the function $F(x)$, apart from the assumption that it belongs to the domain of normal attraction of the stable law $G_\alpha(x)$. Less general estimates are obtained under a small restriction.

The principal source of information for obtaining estimates of the rate of convergence will be for us the following auxiliary proposition, due to B. V. Gnedenko ([3], p. 195):

Lemma 1. *In order that the function $F(x)$ belong to the domain of normal attraction of the stable law $G_\alpha(x)$, it is necessary and sufficient that the following conditions be satisfied:*

$$F(x) = \frac{C_1 a^\alpha + \varphi_1(x)}{|x|^\alpha} \quad \text{for } x < 0,$$

$$1 - F(x) = \frac{C_2 a^\alpha + \varphi_2(x)}{x^\alpha} \quad \text{for } x > 0, \tag{4}$$

where C_1, C_2 are constants, by means of which the constants C and β of the limiting law $G_\alpha(x)$ are determined in a certain way; a has the same meaning as in (1); the functions $\varphi_1(x)$ and $\varphi_2(x)$ are such that $\lim_{x\to-\infty} \varphi_1(x) = \lim_{x\to\infty} \varphi_2(x) = 0$.

We shall give one more auxiliary proposition, due to Esseen:

Lemma 2. Let $F(x)$ be a distribution function with characteristic function $f(t)$, and let $G(x)$ be a function of bounded variation satisfying the conditions $G(-\infty) = 0,\ G(+\infty) = 1$, and having everywhere a bounded derivative $G'(x)$. Suppose that, for some positive constant T and $\varepsilon > 0$,

$$\int_{-T}^{T} \left| \frac{f(t) - g(t)}{t} \right| dt = \varepsilon;$$

then

$$\sup_x |F(x) - G(x)| < A\varepsilon + B/T,$$

where A and B are absolute constants.

Introduce the following notation

$$f(t) = \int e^{itx}\, dF(x), \qquad f_n(t) = \int e^{itx}\, dF_n(x) = \exp\left(-i\frac{t}{B_n} n A_n\right)\left[f\left(\frac{t}{B_n}\right)\right]^n,$$

$$\chi(x) = \frac{|\varphi_1(-x)| + |\varphi_2(x)|}{x^\alpha} \qquad (x > 0),$$

$$\Phi(x) = -\operatorname{Var}_x^\infty \chi(x) = -\int_x^\infty |d\chi(x)| \qquad (x > 0),$$

$$\omega_1(x) = \int_x^\infty \chi(x)\, dx, \qquad L_n = \frac{1}{B_n}\int_0^{B_n} \omega_1(x)\, dx,$$

$$\omega_2(x) = \int_x^\infty x^\alpha\, d\Phi(x), \qquad M_n = \frac{1}{B_n}\int_0^{B_n} \omega_2(x)\, dx,$$

$$\omega_3(x) = x\int_x^\infty d\Phi(x), \qquad N_n = \frac{1}{B_n}\int_0^{B_n} \omega_3(x)\, dx.$$

In these terms the following theorems are formulated:

Theorem 1. If the function $F(x)$ belongs to the domain of normal attraction of the stable law $G_\alpha(x)$ for $1 < \alpha < 2$, then for all sufficiently large n the estimate

$$\sup_x |F_n(x) - G_\alpha(x)| < K_1\left\{n^{(\alpha-1)/\alpha} L_n + \frac{1}{B_n}\right\}.$$

holds.

Theorem 2. If the function $F(x)$ belongs to the domain of normal attraction of the stable law $G_\alpha(x)$ for $0 < \alpha < 1$, then for all sufficiently large n

$$\sup_x |F_n(x) - G_\alpha(x)| < K_2\{N_n \log N_n + n^{-1}\}$$

provided the centering coefficients A_n are chosen in the following way:

$$A_n = \int_0^{B_n} x\, d\left\{\frac{\varphi_1(-x) - \varphi_2(x)}{x^\alpha}\right\}.$$

Theorem 3. If, in the conditions of Theorem 2, one additionally requires the fulfillment of the condition

$$-\int_a^\infty x^\alpha \, d\Phi(x) < +\infty \qquad (a > 0 \text{ arbitrary}),$$

then

$$\sup_x |F_n(x) - G_\alpha(x)| < K_3\{M_n + n^{-1}\}$$

for all sufficiently large n.

Theorem 4. If $F(x)$ belongs to the domain of normal attraction of the stable law $G_\alpha(x)$, for which $\alpha = 1$, $\beta = 0$ (the Cauchy law), then for all sufficiently large n

$$\sup_x |F_n(x) - G_\alpha(x)| < K_4\{N_n \log N_n + n^{-1}\},$$

whereas if, in addition, one requires that the condition

$$-\int_a^\infty x \, d\Phi(x) < +\infty \qquad (a > 0 \text{ arbitrary}),$$

be fulfilled, then the estimate

$$\sup_x |F_n(x) - G_\alpha(x)| < K_5\{M_n + n^{-1}\}.$$

holds.

In the formulations of the theorems K_1, K_2, K_3, K_4, K_5 are constants not depending on n.

An idea of the method used in proving all the theorems stated above is given by

Proof of Theorem 1. Since, under the conditions of Theorem 1, $F(x)$ has a finite mathematical expectation, we may, without restricting generality, put

$$\int x \, dF(x) = \int_0^\infty (1 - F(x) - F(-x)) \, dx = 0, \qquad A_n = 0,$$

which corresponds to the convention adopted earlier that $\gamma = 0$, and, since in this case

$$f(t) = 1 - t\int_0^\infty (1 - F(x) + F(-x)) \sin tx \, dx +$$

$$+it\int_0^\infty (1-F(x)-F(-x))(\cos tx-1)\,dx,$$

taking (2) into account we obtain

$$n\left[f\left(\frac{t}{B_n}\right)-1\right]=\log\varphi_\alpha(t)+\frac{t}{B_n}n\int_0^\infty \frac{\varphi_1(-x)+\varphi_2(x)}{x^\alpha}\,\sin\frac{t}{B_n}x\,dx+$$

$$+i\frac{t}{B_n}n\int_0^\infty \frac{\varphi_2(x)-\varphi_1(-x)}{x^\alpha}\left(\cos\frac{t}{B_n}x-1\right)\,dx.$$

Here, in separating out in the right-hand side the term $\log\varphi_\alpha(t)$, the notations

$$C=(C_1+C_2)\int_0^\infty \frac{\sin z}{z^\alpha}\,dz,\qquad \beta=\frac{C_1-C_2}{C_1+C_2}.$$

have been introduced.

Estimating the integrals on the right, using the known inequalities for trigonometric functions, we obtain

$$\left|n\left[f\left(\frac{t}{B_n}\right)-1\right]-\log\varphi_\alpha(t)\right|\le 2t^2I_n^{(1)}+3|t|I_n^{(2)}, \tag{5}$$

where

$$I_n^{(1)}=\frac{n}{B_n^2}\int_0^{B_n} xX(x)\,dx,\qquad I_n^{(2)}=\frac{n}{B_n}\omega_1(B_n).$$

By virtue of the obvious inequality

$$\left|f\left(\frac{t}{B_n}\right)-1\right|\le\frac{|t|}{B_n}\int_{|t|}|x|\,dF(x)$$

we can choose $\varepsilon_1>0$ such that, for $|t|\le\varepsilon_1\nu_n$, where $\nu_n=\min\{B_n,1/I_n^{(1)},1/I_n^{(2)}\}$, the inequality

$$\left|f\left(\frac{t}{B_n}\right)-1\right|>\frac{1}{2}$$

is satisfied and, as a consequence, the inequality

$$\left|\log f_n(t) - n\left[f\left(\frac{t}{B_n}\right) - 1\right]\right| \le n\left|f\left(\frac{t}{B_n}\right) - 1\right|^2 = \frac{1}{n}\left\{n\left|f\left(\frac{t}{B_n}\right) - 1\right|\right\}^2.$$

Consequently, on the basis of (5), for $|t| < \varepsilon_1 \nu_n$

$$\delta_n = |\log f_n(t) - \log \varphi_\alpha(t)| \le 2t^2 I_n^{(1)} + 3|t| I_n^{(2)} + \frac{1}{n}\left\{n\left|f\left(\frac{t}{B_n}\right) - 1\right|\right\}^2.$$

In connection with the fact that the third term on the right, when estimating the remainder term in (2), gives a term of order $O(1/n)$ (and here (5) is again taken into account), in order to avoid cumbersome notation we shall henceforth discard it, and easily arrive at the conclusion that in the interval $|t| \le \varepsilon_1 \nu_n$

$$|f_n(t) - \varphi_\alpha(t)| \le e^{-C|t|^\alpha} \delta_n e^{\delta_n} =$$

$$= (2t^2 I_n^{(1)} + 3|t| I_n^{(2)}) \exp\{-C|t|^\alpha (1 - 2C^{-1}|t|^{2-\alpha} I_n^{(1)}) + 3|t| I_n^{(2)}\}.$$

It is obvious that the quantity $\varepsilon_1 > 0$ could have been chosen from the very beginning so that, in the interval under consideration, the inequality $1 - 2C^{-1}|t|^{2-\alpha} I_n^{(1)} > 1/2$ would also be fulfilled (which we do). Since $\exp\{3|t| I_n^{(2)}\}$ is bounded for $|t| \le \varepsilon_1 \nu_n$, we obtain the estimate

$$|f_n(t) - \varphi_\alpha(t)| \le D(2t^2 I_n^{(1)} + 3|t| I_n^{(2)}) e^{-\frac{1}{2}C|t|^\alpha},$$

where D is a constant independent of n.

Putting now in Lemma 2

$$F(x) = F_n(x), \qquad G(x) = G_n(x), \qquad T = \varepsilon_1 \nu_n, \qquad \varepsilon = \int_{-T}^{T} \left|\frac{f_n(t) - \varphi_\alpha(t)}{t}\right| dt,$$

we easily arrive at the estimate

$$\sup_x |F_n(x) - G_\alpha(x)| \le K_1 \left\{a(I_n^{(1)} + I_n^{(2)}) + \frac{1}{B_n}\right\}.$$

The validity of Theorem 1 follows from the fact that

$$a(I_n^{(1)} + I_n^{(2)}) = n^{(\alpha-1)/\alpha} L_n.$$

Examples show that the obtained estimates of the rate of convergence to stable laws are essential and, generally speaking, cannot be improved by means of a factor having a power order of smallness. The estimates of Theorems 1, 3, and the second estimate of Theorem 4 cannot be improved even by means of factors tending to zero arbitrarily slowly.

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Received
9 VI 1965

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Note: Figure translations are in progress. See original paper for figures.