THE MONOTONICITY PRINCIPLE IN INTERPOLATION THEORY

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Abstract

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MATHEMATICS

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THE MONOTONICITY PRINCIPLE IN INTERPOLATION THEORY

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1°. The classical theorem of S. N. Bernstein (1)–G. Faber (2) asserts that there exists no matrix of nodes

$$\begin{array}{c} x_1^{(1)} \\ x_1^{(2)}\ x_2^{(2)} \\ \cdots \\ -1 \le x_1^{(n)} < x_2^{(n)} < \cdots < x_n^{(n)} \le 1, \qquad n = 1, 2, \ldots, \end{array} \tag{m}$$

for which, for every function $f(x)$ continuous on the segment $[-1, 1]$,* the uniform relation

$$L_n(f, x) \to f(x), \qquad n \to \infty, \qquad -1 \le x \le 1, \tag{1}$$

holds, where $L_n(f, x)$ is the Lagrange interpolation polynomial of degree $(n-1)$, constructed for the points $\{x_k^{(n)}\}_{k=1}^n$. Therefore, in order to obtain positive results, one has to impose additional restrictions on the matrix of nodes and on the function being interpolated. As such a restriction we have chosen the condition of monotonicity of the matrix of nodes.

We shall say that the matrix of nodes (m) **has the property of monotonicity with weight** $\psi_n(x)$, $n = 1, 2, \ldots$, if there exists a function $\psi_n(x)$, nonzero on the segment $[-1, 1]$,** such that at every point $x \in [-1, 1]$ the following inequalities hold:

$$\text{if } x_k^{(n)} < x_{k+1}^{(n)} \le x, \quad \left|\frac{l_k^{(n)}(x)}{\psi_n(x_k^{(n)})}\right| \le \left|\frac{l_{k+1}^{(n)}(x)}{\psi_n(x_{k+1}^{(n)})}\right|, \quad n = 1, 2, \ldots,$$
$$\text{if } x \le x_k^{(n)} < x_{k+1}^{(n)}, \quad \left|\frac{l_k^{(n)}(x)}{\psi_n(x_k^{(n)})}\right| \ge \left|\frac{l_{k+1}^{(n)}(x)}{\psi_n(x_{k+1}^{(n)})}\right|, \quad n = 1, 2, \ldots,$$

where $\{l_k^{(n)}(x)\}_{k=1}^n$ are the fundamental Lagrange polynomials of the n-th row of the matrix (m).

We shall say that the matrix of nodes (m) is **regular with weight** $\psi_n(x)$, $n = 1, 2, ...$, if it has the property of monotonicity with weight $\psi_n(x)$, $n = 1, 2, ...$, and there exists a constant $C > 0$, independent of x and n, such that

$$\sum_{k=1}^{n} \left[\frac{l_k^{(n)}(x)}{\psi_n(x_k^{(n)})} \right]^2 \leq C, \qquad -1 \leq x \leq 1, \qquad n = 1, 2, \dots .$$

Of special interest is the case when $\psi_n(x) = 1$, $n = 1, 2, ...$ In this case we say that the matrix of nodes is **regular**. The class

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* The set of all such f will be denoted by C.

** If the points $x = \pm 1$ are not nodes, then it is sufficient to require that $\psi_n(x) \neq 0$, $n = 1, 2, ...$, only in the interval $(-1, 1)$.

denote the class of all regular matrices of nodes by K. We indicate some results valid for the class K.

1) If $m \in K$ and $f(x)$ is absolutely continuous on $[-1, 1]$, then relation (1) holds uniformly on $[-1, 1]$ ([3]).
2) If $m \in K$ and $f(x)$ is of bounded variation on $[-1, 1]$, then relation (1) holds at every point of continuity of $f(x)$ ([3]).
3) Let $\{A_n(f, x)\}$ be the interpolation process of S. N. Bernstein ([4]), constructed for a matrix $m \in K$ and any $f \in C$; then ([5])

$$|f(x) - A_n(f, x)| \leq C\omega_f(n^{-1/3}), \tag{2}$$

where ω_f is the modulus of continuity of $f(x)$.

2°. In connection with the results mentioned, the question arises of finding matrices of nodes possessing the monotonicity property.

Theorem 1. *Suppose the points $x = \pm 1$ are not nodes. Then, in order that the system of nodes (m) have the monotonicity property with weight $\psi_n(x)$, $n = 1, 2, ...$, it is necessary and sufficient that the inequalities*

$$\left| \frac{l_1^{(n)}(1)}{\psi_n(x_1^{(n)})} \right| \leq \left| \frac{l_2^{(n)}(1)}{\psi_n(x_2^{(n)})} \right| \leq \dots \leq \left| \frac{l_n^{(n)}(1)}{\psi_n(x_n^{(n)})} \right|, \qquad n = 1, 2, \dots ,$$

$$\left| \frac{l_1^{(n)}(-1)}{\psi_n(x_1^{(n)})} \right| \geq \left| \frac{l_2^{(n)}(-1)}{\psi_n(x_2^{(n)})} \right| \geq \dots \geq \left| \frac{l_n^{(n)}(-1)}{\psi_n(x_n^{(n)})} \right|, \qquad n = 1, 2, \dots .$$

hold.

Thus, in this case, monotonicity need be checked only at the points $x = \pm 1$.

Theorem 2. *Let the matrix of nodes* (m) *consist of the roots of the Jacobi polynomials* $J_n(x) = J_n(x, \alpha_n, \beta_n)$, $n = 1, 2, \ldots$. *In order that this matrix possess monotonicity with weight* $\psi_n(x) = \sqrt{(1-x^2)}\Phi_n(x)$, $n = 1, 2, \ldots$, *where* $\Phi_n(x) > 0$, $-1 < x < 1$, *is a differentiable function, it is sufficient that at each point* $x \in [-1, 1]$ *the inequalities*

$$\max\{x - 1;\, -[1 + \alpha_n - \beta_n + (\alpha_n + \beta_n)x]\} \leq$$

$$\leq v(x) \leq \min\{1 + x;\, 1 + \alpha_n - \beta_n - (\alpha_n + \beta_n)x\}, \tag{3}$$

hold, where $v(x) = (1 - x^2)\Phi'_n(x)/2\Phi_n(x)$.

We outline the proof. Put

$$F(x, y) = \varphi(x)\Phi_n(x)(x - y)^2,$$

$$\varphi(x) = \gamma_n J_n^2(x) + (1 - x^2)(J'_n(x))^2, \qquad \gamma_n = n(n + \alpha_n + \beta_n + 1).$$

With the aid of the differential equation for the Jacobi polynomials we obtain

$$F'_x = (x - y)\{(J'_n(x))^2[2q(x)(x - y)\Phi_n(x) + (1 - x^2)(2\Phi_n(x) +$$

$$+ (x - y)\Phi'_n(x))] + \gamma_n J_n^2(x)(2\Phi_n(x) + (x - y)\Phi'_n(x))\}, \tag{4}$$

where $q(x) = (\alpha_n + \beta_n + 1)x + \alpha_n - \beta_n$. From (4) one can infer that, if the inequalities (3) are satisfied, then the sign of $F'_x(x, y)$, for $(x, y) \in [-1, 1; -1, 1]$, coincides with the sign of the difference $(x - y)$. It remains only to note that at the roots $\{x_k^{(n)}\}_{k=1}^n$ of the Jacobi polynomials the equalities

$$F(x_k^{(n)}, y) = (x_k^{(n)} - y)^2(1 - (x_k^{(n)})^2)(J'_n(x_k^{(n)}))^2\Phi_n(x_k^{(n)}), \qquad k = 1, 2, \ldots, n.$$

hold.

Remark. If $v(x)$ is an odd function, then for Theorem 2 to be valid it is sufficient to verify that only one of the inequalities (3) holds. The second inequality is then satisfied automatically.

Theorem 3. *Let $-1 \le \mu_n \le 0$, $n = 1, 2, ...,$ and let the numbers α_n, β_n satisfy the inequalities*

$$\frac{\mu_n - 1}{2} \le \alpha_n \le \frac{\mu_n + 1}{2}, \qquad \frac{\mu_n - 1}{2} \le \beta_n \le \frac{\mu_n + 1}{2}, \qquad n = 1, 2, \ldots .$$

Then the node matrix (m), composed of the roots of Jacobi polynomials with parameters α_n, β_n, has the monotonicity property with weight

$$\psi_n(x) = (1 - x^2)^{(1+\nu_n)/2}.$$

We outline the proof. By simple computations one verifies that in this case $v(x) = -\mu_n x$. We apply Theorem 2. Since $v(x)$ is an odd function, according to the remark it is enough to check only the fulfillment of the first of inequalities (3).

Corollary 1. If the matrix (m) is composed of the roots of Jacobi polynomials with parameters satisfying the inequalities

$$-1 \le \alpha_n \le 0, \qquad -1 \le \beta_n \le 0, \qquad n = 1, 2, \ldots, \tag{5}$$

then it has the monotonicity property.

This assertion is known (6). It is a special case of Theorem 3 when $\mu_n = -1,\ n = 1, 2, ...$

Corollary 2. If the matrix (m) is composed of the roots of Jacobi polynomials with parameters satisfying the inequalities

$$|\alpha_n| \le \tfrac{1}{2}, \qquad |\beta_n| \le \tfrac{1}{2}, \qquad n = 1, 2, \ldots, \tag{6}$$

then it has the monotonicity property with weight $\psi_n(x) = \sqrt{1 - x^2}$.

This corollary follows from Theorem 3 when $\mu_n = 0,\ n = 1, 2, ...$

With the help of Corollary 2 one can obtain the following theorem.

Theorem 4. Let $x_k^{(n)} = \cos\theta_k^{(n)},\ k = 1, 2, \ldots, n,$ be the roots of the Jacobi polynomial $J_n(x) = J_n(x, \alpha, \beta)$ with parameters satisfying inequalities (6), and let $\{\rho_k^{(n)}\}_{k=1}^n$ be the corresponding Christoffel numbers. Then the inequalities

$$\frac{\rho_k^{(n)} \left|J_{n+1}(x_k^{(n)})\right|}{\sin\theta_k^{(n)} \sin^2\theta_k^{(n)}/2} \le \frac{\rho_{k+1}^{(n)} \left|J_{n+1}(x_{k+1}^{(n)})\right|}{\sin\theta_{k+1}^{(n)} \sin^2\theta_{k+1}^{(n)}/2}, \qquad k = 1, 2, \ldots, (n-1);\ n = 1, 2, \ldots,$$

$$\frac{\rho_k^{(n)}\left|J_{n+1}(x_k^{(n)})\right|}{\sin\theta_k^{(n)}\cos^2\theta_k^{(n)}/2} \geq \frac{\rho_{k+1}^{(n)}\left|J_{n+1}(x_{k+1}^{(n)})\right|}{\sin\theta_{k+1}^{(n)}\cos^2\theta_{k+1}^{(n)}/2}, \qquad k = 1, 2, \dots, (n-1);\ n = 1, 2, \dots.$$

Analogous inequalities for the case when α_n and β_n satisfy conditions (5) are known (7).

Theorem 2 also serves for the proof of the following theorems.

Theorem 5. Let the matrix (m) be composed of the roots of ultraspherical polynomials $J_n(x, \alpha_n, \alpha_n)$, $n = 1, 2, \dots$. Then for any $\alpha_n > \frac{1}{2}$ the matrix (m) has monotonicity in the segment $[-\frac{1}{2}\alpha_n, \frac{1}{2}\alpha_n]$.*

Theorem 6. Let the matrix (m) be composed of the roots of ultraspherical polynomials $J_n(x, \alpha_n, \alpha_n)$, where $|\alpha_n| \leq \frac{1}{2}$. Then it has the monotonicity property with weight $\psi_n(x) = \sqrt{1-x^2}e^{x^2/2}$.

The same assertion remains valid if

$$\psi_n(x) = \sqrt{1-x^2}e^{\pm x^2/2},$$

where the minus sign is taken for $x \leq 0$ and the plus sign for $x > 0$.

3°. The node matrix (m) may have the monotonicity property also in the case when the points $x = \pm 1$, or one of them, are interpolation nodes. By direct verification one can make sure that the node matrices

$$x_k^{(n)} = \cos\frac{n-k}{n}\pi, \qquad k = 0, 1, 2, \dots, n;\ n = 1, 2, \dots, \qquad (m_1)$$

$$x_k^{(n)} = \cos\frac{2(n-k)}{2n+1}\pi, \qquad k = 0, 1, 2, \dots, n;\ n = 1, 2, \dots \qquad (m_2)$$

are regular and, in particular, possess the monotonicity property. Since for regular matrices the inequality (2) holds, it follows that for every $f \in C$ the relation

$$A_n(f, x) \to f(x), \qquad n \to \infty, \qquad -1 \leq x \leq 1, \qquad (7)$$

* If $-1 \leq \alpha_n \leq \frac{1}{2}$, then monotonicity takes place in $[-1, 1]$. By definition,

$$J_n(x, -1, -1) = \int_{-1}^{x} P_{n-1}(t)\,dt,$$

where $P_n(x)$ is the Legendre polynomial of degree n.

holds uniformly, if the polynomials $A_n(f,x)$ are constructed at the nodes (m_1) or at the nodes (m_2). Relation (7) is useful to compare with a theorem of P. Szasz [8], who constructed polynomials $S_n(f,x)$ of degree $2n$, coinciding at the points (m_2) with the function $f(x)$ and converging uniformly on $[-1,1]$ for every $f \in C$. The polynomials $A_n(f,x)$ of S. N. Bernstein have the important property that the ratio λ_n of the degree of the polynomial $A_n(f,x)$ to the number of its nodes can be made arbitrarily close to one, whereas for the polynomials $S_n(f,x)$, $\lambda_n = 2 - 2/(n+1)$.

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Note: Figure translations are in progress. See original paper for figures.