DIFFICULTIES IN MINIMIZING BOOLEAN FUNCTIONS ON THE BASIS OF UNIVERSAL APPROACHES\

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Abstract

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MATHEMATICS

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DIFFICULTIES IN MINIMIZING BOOLEAN FUNCTIONS ON THE BASIS OF UNIVERSAL APPROACHES*

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1. This note is a development of ([2]). On the basis of constructing examples of "bad" functions, the effectiveness is estimated of three presently known universal approaches to the minimization of Boolean functions—the enumeration of irredundant D.N.F.' s, the statistical approach,** and the local approach ([4–6]). It turns out (see item 3) that the difficulties of the above-mentioned universal approaches are combined, and the difficulties for each of them are expressed quite clearly. In particular, it has been found that local algorithms are not algorithms of "bounded laboriousness," despite their "locality." The case of a small number of variables n deserves attention. As follows from Table 1, already for $n \geq 11$ one can specify functions whose minimization is practically infeasible by the known universal methods, while minimization by Quine' s method ([7]), by the method of regular points ([4]), etc., is impossible in principle. The situation is substantially complicated by the fact that "refusal to minimize" *** is inadmissible because of the considerable difference in complexity between irredundant and minimal D.N.F.' s.

2. Let $f(\vec{x}^n)$ be a Boolean function of the variables $x_1, \dots, x_n$; $S(f)$ its reduced D.N.F.; Min f an arbitrary minimal D.N.F. for f; $\Sigma M(f)$ the D.N.F. sum of the minimal ones;**** $t(f)$ the number of irredundant D.N.F.' s of the function f.

First let us recall the notion of neighborhood ([4]). Let K and Q be conjunctions from $S(f)$. We shall call the distance of K from Q the least number r such that in $S(f)$ one can select a chain of $r+1$ conjunctions $P_0, P_1, \dots, P_r$, where $P_0 = Q$, $P_r = K$, and P_i intersects with P_{i+1}, $i = 0, 1, \dots, r-1$. We shall call the neighborhood of order k of a conjunction Q from the D.N.F. $S(f)$, denoted $V_k(Q, S(f))$, the D.N.F. composed of all conjunctions of the D.N.F. $S(f)$ whose distance from Q does not exceed k. We shall call the radius of the function f and denote it by $p(f)$ the least number p such that for any Q from the D.N.F. $S(f)$, $V_p(Q, S(f))$ realizes f.

Next we briefly describe the three approaches mentioned. The starting point is the D.N.F. $S(f)$. Enumeration of irredundant D.N.F.' s consists in constructing all irredundant D.N.F.' s for f and selecting from them Min f by comparing the complexities of these D.N.F.' s; the "difficulty" is measured by the number $t(f)$. The statistical approach consists in constructing "at random" one or several irredundant D.N.F.' s for f, with an estimate of the probability that the D.N.F. Min f will be one of those constructed; the "difficulty"is measured by the fraction constituted by the nonminimal irredundant D.N.F.' s among the total number of irredundant D.N.F.' s for f. The local approach

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* Minimization in the class of disjunctive normal forms (D.N.F.' s) is meant; see ([1]).

** The first two approaches are analogues of the corresponding approaches considered by S. V. Yablonskii ([3]) for contact circuits.

*** That is, restricting minimization to the construction of some one irredundant D.N.F.

**** The D.N.F. $\Sigma M(f)$ is composed of all conjunctions that enter into at least one minimal D.N.F. for f.

is carried out in the form of **local algorithms** ([4–6]). "Locality" manifests itself in the fact that the d.n.f. $S(f)$ is considered not "as a whole," but "in parts," composed of "nearby" conjunctions. A local algorithm A of index k analyzes conjunctions from $S(f)$ by examining neighborhoods of order k of these conjunctions; a set of l properties* of conjunctions, "ascertained" and "remembered" in this analysis, is called the memory of the algorithm A; all conjunctions with respect to which algorithm A has established that they do not belong to $\Sigma M(f)$ are deleted from $S(f)$; the resulting d.n.f. will be denoted by $A(f)$. If for not a single conjunction from $S(f)$ can algorithm A establish whether it belongs to $\Sigma M(f)$, then we shall call the function f **unchanged** by algorithm A. If the d.n.f. $A(f)$ coincides with the original d.n.f. $S(f)$, then we shall call the function f **not simplified** by algorithm A. A function f is not simplified when it is unchanged or when the d.n.f. $\Sigma M(f)$ coincides with $S(f)$. The **complexity** of algorithm A is measured by the number of objects examined in the investigation of a neighborhood ([6]). If $k \geq p(f)$, then A degenerates into an inspection of the irredundant d.n.f.' s of the function f, and the complexity is equal to $t(f)$.

Finally, let us introduce the parameters $R(f)$ and $\delta(f, R)$, characterizing the "relevance" of minimization. Denote by T an arbitrary irredundant d.n.f. for f, and by $Z(T)$ its complexity (number of letters). Put

$$R(T) = Z(T)/Z(\mathrm{Min}\, f), \qquad R(f) = \max_{T \text{ for } f} R(T).$$

Let N be a given number, and let $t(f, N)$ be the number of those T for which

$R(T) > N$. Put $\delta(f,N) = t(f,N)/t(f)$.

3. The purpose of this note is to construct functions $\lambda(\vec{x}^{\,n})$ and $\mu(\vec{x}^{\,n})$ which, for all n, beginning with some n_0, possess the following properties:

	$\lambda(x_1,\dots,x_n)$	$\mu(x_1,\dots,x_n)$
I	(R) $R(\lambda) > 2^{n/8}$	$R(\mu) = 2^{n(1-o(1))}$
I	(T) $t(\lambda) > 2^{2^{n/2}} > 2^{R(\lambda)}$	$t(\mu) > 2^{R(\mu)}$
I	(RT) $\delta(\lambda, 2^{n/10}) \to 1$ as $n \to \infty$	$\delta(\lambda, \frac{1}{2}R(\mu)) > 1 - 2^{-R(\mu)}$
II	(V) $p(\lambda) = 2$ for all n	$p(\mu) = 3$ for all n
II	(L) Unchanged by local algorithms for $k = 1$	Unchanged by local algorithms for $k = 1,\ k = 2$.
II	(LT) The complexity of local algorithms for $k \geq 2$ is equal to $t(\lambda)$	The complexity of local algorithms for $k = 2$ is equal to $2^{R(\mu)}$, and for $k \geq 3$ is equal to $t(\mu)$
III	(Σ) $(\Sigma M)\lambda$ coincides with $S(\lambda)$	
III	(S) All conjunctions of the d.n.f. $S(\lambda)$ have the same length	

Roughly speaking, for these functions the restrictions on "local structure" (II) are combined with the "richness" of the set of irredundant d.n.f.' s (I).

The **main results** are Proposition 4 and Theorems 1 and 2, following from it, on universal approaches to minimization. Let $\{\,{}^n f(\vec{x}^{\,n})\,\}$ be a sequence of functions, c a constant, $c > 1$. We shall call a minimization algorithm A an **algorithm of the type of complete inspection of variants** on $\{\,{}^n f(\vec{x}^{\,n})\,\}$ if, for all n, beginning with some n_0, the algorithm A includes the inspection of all irredundant d.n.f.' s of the function ${}^n f$ and if $t({}^n f) \geq c^{c^r}$. (In this situation the complexity of inspecting irredundant d.n.f.' s and the complexity of the algorithm A are inadmissibly large–

* Examples of properties: "the conjunction Q from $S(f)$ belongs to $\Sigma M(f)$," "Q belongs to all minimal d.n.f.' s for f," etc. In ([5,6]) twelve such properties are singled out, representing independent interest ("basic predicates"). We shall stipulate that for $l \leq 12$, the memory consists only of the basic predicates. For all practically used local algorithms $k \leq 2,\ l \leq 2$ ([6]).

···ones.) We shall call $\{{}^n f(\vec{x}^{\,n})\}$ **actual** if $\delta({}^n f, c^n) \to 1$ as $x \to \infty$*. (In this situation the "statistical approach" is ineffective.)

Theorem 1. *For any local algorithm A, for any choice of its parameters k and l, $k \neq 1$, one can specify a sequence $\{{}^n f(\vec{x}^{\,n})\}$ on which A is an algorithm of the type of complete enumeration of variants; moreover, for all A one can specify one and the same sequence.*

Theorem 2. *Theorem 1 remains valid if one restricts consideration only to actual sequences.*

4. Construction of the functions $\lambda(\vec{x}^{\,n})$ and $\mu(\vec{x}^{\,n})$. For simplicity we shall assume that $n = 4p$, $p = 1, 2, \ldots$. We divide the variables $x_1, \ldots, x_n$ into p groups of 4 variables each: $x_{i1}, x_{i2}, x_{i3}, x_{i4}$, $i = 1, 2, \ldots, p$. Put $\varphi(x, y, z, w) = 0$ when $x = y = z = w$, and $\varphi(x, y, z, w) = 1$ in the remaining cases,

$$\lambda(\vec{x}^{\,n}) = \&_{i=1}^{p} \varphi(x_{i1}, x_{i2}, x_{i3}, x_{i4}).$$

By induction on $r = 1, 2, \ldots$ we construct the functions $\pi^r(\vec{u}^{\,k})$ (2). Put

$$\pi(x, y, z, w) = xz\bar{w} \vee \bar{x}\bar{z}w \vee y\bar{z}\bar{w} \vee \bar{y}zw,$$

$$L(x^s) = x_1 + \cdots + x_s \pmod 2.$$

We agree that $\pi^0(\vec{u}^{\,k}) = L(\vec{u}^{\,k})$. We divide the variables $u_1, \ldots, u_k$ into three groups: $x_1, \ldots, x_l$, $y_1, \ldots, y_m$, and z, w, $l + m + 2 = k$, $|l - m| \leq 1$. If the functions $\pi^{r-1}(\vec{x}^{\,l})$, $\pi^{r-1}(\vec{y}^{\,m})$ have already been constructed, then let

$$\pi^r(\vec{u}^{\,k}) = \pi(\pi^{r-1}(\vec{x}^{\,l}), \pi^{r-1}(\vec{y}^{\,m}), z, w).$$

Table 1

	Parameters	${}^{11}f$	${}^{13}f$	${}^{15}f$	${}^{17}f$	${}^{19}f$	${}^{21}f$	${}^{23}f$	${}^{25}f$	${}^{35}f$
1	$R = R({}^n f)$	4	8	17	40	77	560	1200	2500	$6 \cdot 10^8$
2	$t({}^n f)$	10^{77}	10^{300}	$t({}^n f) \geq 10^{2000}$	$t({}^n f) \geq 10^{2000}$	$t({}^n f) \geq 10^{2000}$	$t({}^n f) \geq 10^{2000}$	$t({}^n f) \geq 10^{2000}$	10^5	10^8
3	$\delta({}^n f, R/2)$	–	–	$1 - \frac{1}{10^{19}}$	$1 - \frac{1}{10^{100}}$	$1 - \frac{1}{10^{80}}$	$1 - \frac{1}{10^{20}}$	$1 - \frac{1}{10^{70}}$	$1 - \frac{1}{10^{100}}$	$1 - \frac{1}{10^{1000}}$
4	$k = 1$ **	Unchanged	Unchanged	Unchanged	Unchanged	Unchanged	Unchanged	Unchanged	Unchanged	Unchanged

	Parameters	^{11}f	^{13}f	^{15}f	^{17}f	^{19}f	^{21}f	^{23}f	^{25}f	^{35}f
4	$k = 2$ **	10^{18}	10^{35}	10^{140}	$\leq$ complexity $\leq$	$\leq$ complexity $\leq$	$\leq$ complexity $\leq$	$\leq$ complexity $\leq$	11^{10^3}	10^{10^5}
4	$k \geq 3$	Degenerates into enumeration of irredundant D.N.F.	Degenerates into enumeration of irredundant D.N.F.	Degenerates into enumeration of irredundant D.N.F.	Degenerates into enumeration of irredundant D.N.F.	Degenerates into enumeration of irredundant D.N.F.	Degenerates into enumeration of irredundant D.N.F.	Degenerates into enumeration of irredundant D.N.F.	Degenerates into enumeration of irredundant D.N.F.	Degenerates into enumeration of irredundant D.N.F.

* Invariance has been proved for the case $l \leq 12$ (see the footnote on p. 14).

Next, we divide the variables $v_1, \dots, v_n$ into two groups: x, y, z, w and $u_1, \dots, u_k$, $k + 4 = n$. Put

$$\mu^r(\vec{v}^{\,n}) = \pi(x, y, z, w) \mathbin{\&} \pi^r(\vec{u}^{\,k}), \qquad \mu(\vec{x}^{\,n}) = \mu^r(\vec{x}^{\,n})$$

for $r = [\frac{1}{2}\log_2 n]$.

Proposition. *The functions $\lambda(\vec{x}^{\,n})$ and $\mu(\vec{x}^{\,n})$ possess, respectively, the properties (R)—(S) and (R)—(LT)**.*

Such a combination of properties is new. Yu. I. Zhuravlev introduced functions ${}^n\psi(\vec{x}^{\,n})$ of the "chains" and "cycles" type and proved that for $k \cdot l <$

* In other words, "almost all" irredundant D.N.F. of the function ${}^n f$ are more complex than the D.N.F. Min ${}^n f$ by a factor of c^n.

** Property (L) has been proved for local algorithms for which $l \leq 2$ (see the footnote on p. 14). Apparently, (L) is valid for any l. For finding a minimal D.N.F. of the function λ, the question of (L) is irrelevant: by virtue of ΣM, the function λ cannot be simplified for any k and l.

$< p({}^n\psi)$, they are invariant under local algorithms of index k with memory l ([5], [6]). However, the complexity of these algorithms on the functions ${}^n\psi$ is "small" —it does not exceed 2^k for all n; moreover, $R({}^n\psi) < 2$ for all n. Further, in ([2]) functions π^r and ξ^r with "large" R and t are constructed, but the properties $(RT) - (LT)$ either are insufficiently expressed or have not been investigated.

Finally, V. V. Glagolev proved that for "almost all" functions of n variables $\log_2 t(f) \geqslant 2^{n(1-o(1))}$, but for them $R < \log_2 n \cdot \log_2 \log_2 n$ ([8], [9]).

5. Table 1 contains lower estimates for functions ${}^n f$ of n variables, where ${}^n f = \mu^1(\nu_1, \dots, \nu_n)$ for $n = 11, 13, \dots, 19$ and ${}^n f = \mu^2(\nu_1, \dots, \nu_n)$ for $n = 21, 23, 25, 35$. Rows 1 and 3 characterize the "relevance" of minimizing the functions ${}^n f$. Rows 2, 3, and 4 characterize the difficulties, respectively, of enumerating irredundant D.N.F.' s, of the statistical approach, and of the local approach. We note the rapid growth of the difficulties with the growth of the number of variables.

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Note: Figure translations are in progress. See original paper for figures.