QUASIANALYTIC FUNCTIONALS IN QUANTUM FIELD THEORY

MATHEMATICAL PHYSICS

1966

SovietRxiv

Abstract

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UDC 517.433

MATHEMATICAL PHYSICS

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QUASIANALYTIC FUNCTIONALS IN QUANTUM FIELD THEORY

(Presented by Academician N. N. Bogolyubov on 29 IX 1965)

In the present note a complete investigation is given of the moment problem in quantum field theory [1–3]. This problem is closely connected with the question of the self-adjoint properties of field operators. It turns out that, in the case of quasianalytic functionals, the field operators admit self-adjoint closures. Moreover, it is shown that the equality $E_\lambda(f)E_\mu(g) = E_\gamma(f \times g)$ holds for the spectral families $E_\lambda(f)$, $E_\lambda(g)$, $E_\gamma(f \times g)$ of the operators $\overline{A(f)}$, $\overline{A(g)}$, $\overline{A(f \times g)}$, respectively. This thereby gives a justification of Neumann rings in field theory. Since our considerations do not depend on the type of basic functions, all the results are valid, in particular, also for local rings of field operators. The results proposed may be applied to the study of self-adjoint extensions and other operators in field theory, and also in statistical physics [4]. For the proof we use methods developed in differential and difference equations [5,6∗].

1. Consider the sequence of spaces $\Phi_0,\ \Phi_4, \dots, \dots, \Phi_{4n}, \dots$, where by Φ_0 is denoted the space of complex numbers, and by Φ_{4n} some space of basic functions of $4n$ variables (for example, the space S_{4n} or $K_{4n}\{M_p\}$). With the aid of this sequence of spaces a system Σ of sequences is constructed,

$$f = \{f_0, f_1(x_1), \dots, f_n(x_1, \dots, x_n), \dots\}, \qquad f_n(x_1, \dots, x_n) \in \Phi_{4n},$$

with multiplication

$$f \times g = \{\dots, (f \times g)_n, \dots\},$$

$$(f \times g)_n = \sum_{i+j=n} f_i(x_1, \dots, x_i) g_j(x_{i+1}, \dots, x_{i+j})$$

and involution

$$f^* = \{\overline{f_0}, \overline{f_1(x_1)}, \dots, \overline{f_n(x_n, \dots, x_1)}, \dots\},$$

where the bar denotes complex conjugation.

With the aid of the initial topology of the spaces Φ_{4n}, the system Σ can be made into a topological space. The space conjugate to it will be denoted by Σ'.

Consider all those elements $f \in \Sigma$ for which

$$\sum_{n=0}^{\infty} \rho_n \int \cdots \int |f_n(x_1, \dots, x_n)|^2 \, dx_1 \cdots dx_n < \infty,$$

where ρ_n is some nondecreasing sequence of numbers greater than one. For such $f \in \Sigma$ we introduce the scalar product

$$\langle f, g \rangle_\rho = \sum_{n=0}^{\infty} \rho_n \int \cdots \int \overline{f_n(x_n, \dots, x_1)} \, g_n(x_1, \dots, x_n) \, dx_1 \cdots dx_n.$$

* As Yu. M. Berezanskii kindly informed the author, he also obtained analogous results for the particular case of the operator $A(f)$, when $f = \{0, f_1(x_1), 0, \dots\}$.

Completing by the usual methods, we obtain the Hilbert space $\mathcal{L}_2(\rho)$. In the case when ρ is the unit sequence, $\mathcal{L}_2(\rho)$ becomes $\mathcal{L}_2(I)$, and $\mathcal{L}_2(I) \supset \mathcal{L}_2(\rho)$, since $\|f\|_\rho \geq \|f\|_I$. The totality of functionals over $\mathcal{L}_2(\rho)$ also forms a certain Hilbert space $\mathcal{L}_2'(\rho^{-1})$ with scalar product $\langle \alpha, \beta \rangle_{\rho^{-1}}$, which, in the case when $\alpha \in \mathcal{L}_2'(\rho^{-1})$, $\beta \in \mathcal{L}_2(\rho)$, coincides with the scalar product $\langle \alpha, \beta \rangle_I$. The inclusions $\mathcal{L}_2(\rho) \subseteq \mathcal{L}_2(I) \subseteq \mathcal{L}_2'(\rho^{-1})$ hold.

We shall denote by $\bar{W}(f)$ all those functionals from Σ' for which the infinite system of inequalities holds

$$\sum_{m,n=0}^{N} \bar{\alpha}_n W_{m+n}(\bar{f}_n f_m) \alpha_m \geq 0,$$

where $W_n(f_n) \in B'_{un}$ and α_n are complex numbers, and, in addition,

$$\sum_{n=0}^{\infty} \frac{|W_{2n}(f_n)|}{\rho_n} < \infty. \tag{1}$$

In the case when the sequence ρ_n defines a quasianalytic class $C(\sqrt{\rho_n} = m_n)$ (7), we shall call the functionals $W(f)$ quasianalytic. The quasianalyticity condition of the class $C(m_n)$ is written in the form

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{m_n}} = +\infty$$

or in the form of Carleman' s criterion (a necessary and sufficient condition)

$$\int_1^{\infty} \frac{\ln T(r)}{r^2} \, dr = +\infty, \qquad \text{where} \qquad T(r) = \sup_{n=0,1,\dots} \frac{r^n}{m_n}.$$

According to (1), $W(f) \in \mathcal{L}_2'(\rho^{-1})$. Here we make no additional assumptions connected with the axioms of field theory, since they will not be needed.

With the aid of the functional $W(f)$ we single out the set

$$\Omega = \{f \in \mathcal{L}_2(\rho) \mid W(f^* \times f) = 0\}.$$

Suppose that degeneracy is absent, i.e. that Ω contains not a single element distinct from zero. Then, with the aid of $W(f)$, we may introduce the scalar product

$$(f, g) = W(f^* \times g), \qquad f, g \in \mathcal{L}_2(\rho) \tag{2}$$

and with it the Hilbert space H. In the case when Ω is nonempty, the formulations of the theorems remain unchanged.

We also note that, in our considerations, the functional $W_n(f_n)$ is taken in the form ([1])

$$W_n(f_n) = \int \cdots \int W_n(x_1, \ldots, x_n) f_n(x_1, \ldots, x_n)\, dx_1 \cdots dx_n.$$

2. For our purposes it is enough to consider the operator of multiplication of elements of Σ by the element

$$f_n = \{0, \ldots, 0, f_n(x_1, \ldots, x_n), 0, \ldots\},$$

where $f_n(x_1, \ldots, x_n)$ is a real function from B_{4n}. If this operator is denoted by $A(f_n)$, then

$$A(f_n)g = f_n \times g =$$
$$= \{0, \ldots, 0, f_n(x_1, \ldots, x_n)g_0, \ldots, f_n(x_1, \ldots, x_n)g_m(x_1, \ldots, x_m), \ldots\}.$$

This operator is symmetric in the scalar product (2) by virtue of the reality of $f_n(x_1, \ldots, x_n)$. Since it has equal defect numbers, its closure $\overline{A(f_n)}$ admits self-adjoint extensions. We shall show that the closure $\overline{A(f_n)}$ is maximal.

Theorem 1. *The closure of the operator $A(f_n)$ in the Hilbert space H generated by the quasi-analytic functional $W(f)$ is a self-adjoint operator.*

The proof of this theorem is carried out with the aid of the following lemma.

Lemma. *Let us consider, in the Hilbert space $\mathcal{L}_2(\rho)$, weak solutions of the equation*

$$du(t)/dt - (zA(f_n))^* u(t) = 0, \qquad 0 \le t < \infty, \tag{3}$$

where z is a fixed complex number. Uniqueness of the Cauchy problem for such solutions holds if and only if the class $C(1/\sqrt{\rho_n})$ is quasi-analytic.

A weak solution of equation (3) is a vector-valued function $u(t)$ with values in $\mathcal{L}_2(\rho)$ for each t, $0 \le t < \infty$, weakly differentiable in t, and satisfying the equality

$$\langle du(t)/dt, f\rangle_\rho = \langle u(t), (zA(f_n))f\rangle_\rho, \qquad f \in D(A(f_n)).$$

From this equality one derives the recurrence relation

$$(d/dt)jv_k(t;x_1,\dots,x_k) = (\bar{z})jv_{k+n}[t;y_1,\dots,y_n;\dots;x_1,\dots,x_k],$$

where $\rho_k u_k(t;x_1,\dots,x_k) = v_k(t;x_1,\dots,x_k)$.

For uniqueness of weak solutions of the Cauchy problem for equation (3), it is enough to prove the property: every weak solution $u(t)$ equal to zero at $t=0$ is identically zero. Therefore consider one such solution $u(0) = u^{(0)} = \{0\}$. Since $\rho_k u_k(t;\dots) = v_k(t;\dots)$, it follows that $v(0) = v^{(0)} = \{0\}$. Therefore, by virtue of the condition $v_k(0) = 0$, we obtain

$$(d/dt)_k^{jv}(t;x_1,\dots,x_k)\big|_{t=0} = 0, \qquad \max_{0\le t<\infty} |v_k(t;x_1,\dots,x_k)| < \sqrt{\rho_k}C_T,$$

$$j = 0,1,2,\dots; \qquad C_T < \infty.$$

Now, putting $\rho_k = m_k$, by virtue of the quasi-analyticity of the class $C(1/\sqrt{m_n})$, the function $v_k(t;x_1,\dots,x_k) \equiv 0$, $k = 0,1,2,\dots$

With the aid of the results obtained, the question of determinacy of the moment problem (1^{-3}) can be investigated completely. The moment problem is called determinate if, in the representation

$$W_n(f_1) = \int_{-\infty}^{\infty} \lambda^n \, d\sigma(\lambda; f_1)$$

the measure $\sigma(\lambda; f_1)$ is unique.

Theorem 2. *If the functional $W(f)$ is quasi-analytic, then the moment problem is always determinate.*

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Received
25 IX 1965

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Note: Figure translations are in progress. See original paper for figures.