ESTIMATE OF THE CARDINALITY OF THE SET OF NONRIGID SLEEVE CONNECTIONS FOR SURFACES OF REVOLUTION

MATHEMATICS

1966

SovietRxiv

Abstract

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UDC 513.73

MATHEMATICS

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ESTIMATE OF THE CARDINALITY OF THE SET OF NONRIGID SLEEVE CONNECTIONS FOR SURFACES OF REVOLUTION

(Presented by Academician I. N. Vekua on November 19, 1965)

Let F be a piece of a surface of revolution of positive curvature, bounded by a parallel. We subject the surface F, under an infinitely small bending, to a special sleeve connection Σ. We shall estimate the cardinality of the set of nonrigid sleeve connections of the surface F. In this formulation, this problem was considered by I. N. Vekua, as a special case of a more general problem, in the monograph (^{1}). It follows from the results of that work that a nonrigid sleeve connection is an exceptional phenomenon. Thus, for example, such limits are indicated for the variation of the angle $\theta(s)$ between the surface and the sleeve that, if all generators of the boundary strip of the surface Σ along the edge F are situated in the indicated region, then the sleeve connection under consideration will certainly be rigid ((^{1}), Theorem 5.20). In the opposite case the sleeve connection may turn out to be nonrigid. However, the picture of the distribution of nonrigid sleeves is given in (^{1}) only for sleeve connections forming a constant angle $\theta(s) = \text{const}$ with the surface. It is proved that such nonrigid sleeves are countable in number and that they converge to the orthogonal one ((^{1}), Theorem 5.21). In the present work another approach to the solution of the stated problem is proposed and, in connection with this, results are obtained that generalize those indicated. Their precise formulation is given below. Here we shall only note that the cardinality of nonrigid sleeve connections is the continuum and that in a neighborhood of the orthogonal sleeve the nonrigid sleeve connections are everywhere dense. In content and methods of investigation, this work is a direct continuation of our note (^{3}). Related questions are also considered in (^{2}).

1. Let us first consider, in the rectangular Cartesian coordinate system $Oxyz$, a surface of revolution F given by the equation $z = f(x^2+y^2)$ and situated in the half-space $z \geq 0$, convex downward. We shall assume that the Gaussian curvature of the surface is positive, i.e. $K \geq k_0 > 0$. Let, further, the surface F be bounded by the parallel L, whose projection onto the plane Oxy is the circle $\Gamma : x^2 + y^2 = 1$. We denote the region $x^2 + y^2 \leq 1$ by D.

We derive the equation of infinitely small bendings of the surface F analogously to how this was done in our work (^{2}). Denote the components of the vector $\mathbf{U}$ of the displacement field of points of the surface under its infinitely small bending by $\{\xi(x,y);\eta(x,y);\zeta(x,y)\}$. Then, as is known, the functions $\lambda = \xi + \dfrac{\partial f}{\partial x}\zeta$ and $\mu = \eta + \dfrac{\partial f}{\partial y}\zeta$ satisfy the system of differential equations of elliptic type

$$\lambda_x = r\zeta; \qquad \lambda_y + \mu_x = 2s\zeta; \qquad \mu_y = t\zeta, \tag{1}$$

where $r = 2f' + 4x^2 f''$, $s = 4xyf''$, $t = 2f' + 4y^2 f''$ (the prime denotes differentiation with respect to the argument). We write system (1) in complex form, putting $z = x + iy$, $w(z) = \lambda + i\mu$. We have

$$\partial_{\bar z} w(z) = 2z^2 f''\zeta; \qquad \mathrm{Re}\{\partial_z w(z)\} = (2f' + 2\rho^2 f'')\zeta, \tag{2}$$

where $\rho^2 = x^2 + y^2$; $2f' + 2\rho^2 f'' \geq c_0 > 0$; $c_0 = \text{const}$. It is easy to see from (2) that the function $w(z)$ satisfies a certain Beltrami equation $\partial_{\bar z} w + q(z)\partial_z w = 0$, where $q(z)$ has a zero of second order at the point $z = 0$. Therefore, by Theorem 2.5 of the book (^{1}), the function $w(z)$ admits the representation

$$w(z) = w(0) + zw_1(z), \tag{3}$$

where $w_1(z)$ has the same smoothness as $w(z)$. Denoting $w_1(z) = U + iV$, we rewrite system $(2)'$ in the form

$$U_x - U_y = 4xf''\zeta; \qquad U_y + V_x = 4yf''\zeta; \qquad xU_x + yU_y + U = 2(f' + 2\rho^2 f'')\zeta. \tag{4}$$

Eliminate the unknown function V from system (4). To do this, we differentiate the first equation (4) with respect to x, the second with respect to y, and add. We finally obtain:

$$\sum_{i,k=1}^{2} \frac{\partial}{\partial x_k}\left(a_{ik}\frac{\partial U}{\partial x_i}\right) + \sum_{i=1}^{2} e_i \frac{\partial U}{\partial x_i} + cU = 0, \tag{5}$$

where $a_{ik} = a_{ki}, e_i, c$ are known functions, determined by the formulas

$$\begin{gathered} a_{11} = \mu(\rho^2) + y^2\lambda(\rho^2); \qquad a_{21} = a_{12} = -xy\lambda(\rho^2); \qquad a_{22} = \mu(\rho^2) + x^2\lambda(\rho^2); \\ e_1 = -x\lambda(\rho^2); \qquad e_2 = -y\lambda(\rho^2); \qquad c = e_{1x} + e_{2y}; \\ \mu(\rho^2) = f'/(f' + 2\rho^2 f''); \qquad \lambda(\rho^2) = 2f''/(f' + 2\rho^2 f''). \end{gathered} \tag{6}$$

We shall show that equation (5), using formulas (6), can be reduced to self-adjoint form. To this end, in (5) we make a change of the unknown function, putting $U = \varphi U_1$, where φ is, for the moment, an undetermined function. We have

$$\begin{gathered}(\varphi a_{11}U_{1x} + \varphi a_{12}U_{1y})_x + (a_{21}\varphi U_{1x} + a_{22}\varphi U_{1y})_y + (a_{11}\varphi_x + a_{12}\varphi_y + \varphi e_1)U_{1x} + \\ (a_{21}\varphi_x + a_{22}\varphi_y + \varphi e_2)U_{1y} + \{(a_{11}\varphi_x + a_{12}\varphi_y)_x + (a_{21}\varphi_x + a_{22}\varphi_y)_y + \\ +(e_1\varphi_x + e_2\varphi_y) + c\varphi\}U_1 = 0.\end{gathered} \tag{7}$$

Choose the function φ so that

$$a_{11}\varphi_x + a_{12}\varphi_y + \varphi e_1 = 0; \qquad a_{21}\varphi_x + a_{22}\varphi_y + \varphi e_2 = 0. \tag{8}$$

We shall show that system (8) is completely integrable if its coefficients are computed from formulas (6). For this, it is evidently sufficient to show that

$$\left[\frac{-e_1a_{22} + e_2a_{12}}{a_{11}a_{22} - a_{12}^2}\right]_y = \left[\frac{e_1a_{12} - a_{11}e_2}{a_{11}a_{22} - a_{12}^2}\right]_x. \tag{9}$$

But, according to formulas (6),

$$\begin{aligned} a_{11}a_{22} - a_{12}^2 &= \mu(\rho^2)[\mu(\rho^2) + \rho^2\lambda(\rho^2)]; \\ e_1a_{22} - e_2a_{12} &= -x\lambda(\rho^2)[\mu(\rho^2) + \rho^2\lambda(\rho^2)]; \\ -e_1a_{12} + e_2a_{11} &= -y\lambda(\rho^2)[\mu(\rho^2) + \rho^2\lambda(\rho^2)]. \end{aligned} \tag{10}$$

Therefore, substituting (10) into expression (9), we obtain the identity

$$[x\lambda(\rho^2)/\mu(\rho^2)]_y = [y\lambda(\rho^2)/\mu(\rho^2)]_x.$$

Then from (8) we find:

$$\varphi = f'. \tag{11}$$

It remains to compute the coefficient of U_1 in equation (7). We have

$$\begin{aligned} &(a_{11}\varphi_x + a_{12}\varphi_y)_x + (a_{21}\varphi_x + a_{22}\varphi_y)_y + e_1\varphi_x + e_2\varphi_y + c\varphi = \\ &= -(\varphi e_1)_x - (\varphi e_2)_y + e_1\varphi_x + e_2\varphi_y + \varphi(e_{1x} + e_{2y}) = \\ &= -(\varphi e_1)_x - (\varphi e_2)_y + (\varphi e_1)_x + (\varphi e_2)_y \equiv 0. \end{aligned}$$

Thus, equation (7) takes the form:

$$\sum_{i,k=1}^{2} \frac{\partial}{\partial x_k}\left(\overline{a}_{ik}\frac{\partial U_1}{\partial x_i}\right) = 0, \qquad \text{where} \quad \overline{a}_{ik} = f' a_{ik}. \tag{12}$$

Equation (12), which has a self-adjoint form, will be used by us in what follows. The relation between the solutions of equation (12) and the trivial infinitesimal bendings of the surface F is given by the following theorem.

Theorem 1. *Every linearly independent infinitesimal trivial bending generates the following solution of equation (8), and conversely: a) rotation about the axis* Oz *and parallel translation along the axes* Ox *and* Oy *generate* $U_1(x, y) \equiv 0$*; b) rotation about the axis* Ox *generates*

$$U_1(x, y) = \frac{f + \rho^2 f'}{\rho^2 f'}\,\Omega_x y;$$

c) rotation about the axis Oy *generates*

$$U_1(x, y) = \frac{f - \rho^2 f'}{\rho^2 f'}\,\Omega_y x;$$

d) parallel translation along the axis Oz *generates* $U_1(x, y) = c_z$*, where* Ω_x, Ω_y, c_z *are arbitrary real constants.*

Remark. If the surface of revolution is projected onto the plane Oxy nonuniquely, then, in order to find its infinitesimal bendings, one uses equation (12), after first subjecting the surface to a projective transformation.

II. 2. We proceed to the derivation of the analytic form of the sleeve constraint. Let us subject the surface F to a nonspecial sleeve constraint Σ. Denote by $\mathbf{n}_\Sigma$ the normal to Σ along L. The orientation of the normal $\mathbf{n}_\Sigma$ is chosen arbitrarily at some point of the edge L, and then, by continuity, is extended to the whole contour L. Next consider an orthogonal sleeve of the surface F and, analogously, choose on it the normal $\mathbf{n}$. Denote by α the angle between $\mathbf{n}$ and $\mathbf{n}_\Sigma$; the angle α is measured from $\mathbf{n}$ to $\mathbf{n}_\Sigma$ counterclockwise, if one looks from the side of the positive direction of the curve L. Evidently, $\alpha = \alpha(s)$, where s is the length of the contour L. Without loss of generality, we assume $0 < \alpha < \pi$.

Introduce into consideration the angle γ, defined by the formula $\gamma = \pi/2 - \alpha - \operatorname{arctg} \partial f/\partial\rho$. Then the coordinates of the vector $\mathbf{n}_\Sigma$ have the form $\{\sin\gamma \cdot x;\ \sin\gamma \cdot y;\ \cos\gamma\}$, and the condition of the sleeve constraint can be written as

$$\sin\gamma\,(x\xi + y\eta) + \cos\gamma \cdot \zeta = 0. \tag{13}$$

Without loss of generality, we assume $\rho = 1$. Introduce into (13) the function $U(x, y)$, analogously to how this was done in (2):

$$xU_x + yU_y + \left[1 + \frac{\sin\gamma \cdot 2(f' + 2f'')}{\cos\gamma - 2f' \sin\gamma}\right] U = \sigma(s). \tag{14}$$

Here the function $\sigma(s)$ is known and depends on two arbitrary constants. Passing in (14) to the required function U_1,

$$f(xU_{1x} + yU_{1y}) + \left[f' + 2f'' + \frac{2f' \sin\gamma\,(f' + 2f'')}{\cos\gamma - 2f' \sin\gamma}\right] U_1 = \sigma(s). \tag{15}$$

Transform the expression in square brackets:

$$f' + 2f'' + \frac{2f' \sin\gamma\,(f' + 2f'')}{\cos\gamma - 2f' \sin\gamma} = (f' + 2f'')\frac{\cos\gamma}{\cos\gamma - 2f' \sin\gamma}.$$

Since $\operatorname{tg}\gamma = (1 - 2f' \operatorname{tg}\alpha)/(2f' + \operatorname{tg}\alpha)$, it follows that

$$\frac{\cos\gamma}{\cos\gamma - 2f' \sin\gamma} = \frac{2f' + \operatorname{tg}\alpha}{2f' + \operatorname{tg}\alpha - 2f' + 4f'^2 \operatorname{tg}\alpha} = \frac{1}{1 + 4f'^2} + \frac{2f'}{1 + 4f'^2} \operatorname{ctg}\alpha.$$

Taking the last equality into account, we rewrite boundary condition (15) as follows:

$$\frac{\partial U_1}{\partial \rho} + \left[\frac{f' + 2f''}{f'(1 + 4f'^2)} + \frac{2(f' + 2f'') \operatorname{ctg}\alpha}{1 + 4f'^2}\right] U_1 = \sigma_1(s), \tag{16}$$

where $\sigma_1 = \sigma/f'$. Since $f' + 2f'' > 0$, denoting the terms in square brackets by $a^2(s)$ and $b^2(s)$, we finally rewrite condition (16) in the form

$$\partial U_1/\partial\rho + [a^2(s) + \operatorname{ctg}\alpha\, b^2(s)]U_1 = \sigma_1(s). \tag{16'}$$

Condition (16′), with respect to equation (12), has a self-adjoint form.

Let us note that $a^2(s) + \operatorname{ctg}\alpha\, b^2(s) > 0$ if

$$\operatorname{ctg}\alpha > -\frac{(f' + 2f'')(1 + 4f'^2)}{f'(1 + 4f'^2)2(f' + 2f'')} = -\frac{1}{2f'}$$

or

$$0 < \alpha < \pi - \operatorname{arctg} \partial f/\partial\rho. \tag{17}$$

§ 3. Theorem 2. *Let the surface $F \in C^3_\alpha$, $0 < \alpha < 1$, be subject to a sleeve constraint Σ_1, determined by the angle $\alpha_1(s)$. Then among the sleeve constraints Σ_ε, determined by the angle $\alpha_\varepsilon(s)$ from the relation $\operatorname{tg}\alpha_\varepsilon(s) = \varepsilon \operatorname{tg}\alpha_1(s)$, where ε is a parameter, $\varepsilon \in (-\infty, 0)$, there exists an exactly countable set of non-rigid constraints. The latter converge to the orthogonal sleeve.*

Proof. Fix the point $(0, 0)$ on the surface together with the tangent plane and subject F to an infinitesimal bending governed by a sleeve constraint. In view of what was said above, the problem of finding nontrivial bendings reduces to finding nontrivial solutions of the problem:

$$\sum_{i,k=1}^{2} \frac{\partial}{\partial x_k}\left(\bar{a}_{ik}\frac{\partial U_1}{\partial x_i}\right) = 0 \quad \text{in } D; \qquad \frac{\partial U_1}{\partial \rho} + (a^2 + \varepsilon^{-1}\operatorname{ctg}\alpha_1 \cdot b^2)U_1 = 0 \quad \text{on } \Gamma. \tag{18}$$

Applying the variational principle (^{4}), one can prove the solvability of problem (18) precisely for an even number of values ε_k^{-1}. The eigenvalues ε_k^{-1} of this problem have no limit point at finite distance, and therefore

$$\alpha_{\varepsilon_k}(s) \to \pi \quad \text{as } \varepsilon_k^{-1} \to \infty.$$

The following assertions follow immediately from Theorem 2.

Theorem 3. *The set of non-rigid sleeve constraints of a surface of revolution bounded by a parallel has the cardinality of the continuum.*

Theorem 4. *For surfaces of revolution bounded by a parallel, there exists a family of non-rigid sleeves depending continuously on a parameter.*

I take this opportunity to express my gratitude to A. P. Morozov for his attention to this work.

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Received
9 XI 1965

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