RELATIONS OF THE TYPE OF THE “LENGTH AND AREA PRINCIPLE”

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Abstract

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RELATIONS OF THE TYPE OF THE "LENGTH AND AREA PRINCIPLE"

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In the study of conformal, quasiconformal, and more general mappings of plane domains, one inequality is very useful, relating areas and lengths and expressing the so-called "length and area principle." In the present work we give certain more general relations of this type for mappings having generalized first partial derivatives with respect to x and y and a finite Dirichlet integral. We also indicate one sufficient condition, following from these relations, for a set situated on the boundary of a domain to be mapped into a set of length zero.

Let F be a closed set in Euclidean n-dimensional space R_n, $F \neq R_n$, and let $r(x)$ be the distance from the point $x \in R_n$ to F. Denote by E_t the set $\{x : r(x) = t\}$, the level set of the function $r(x)$.

The proof of the relations (3)–(6) given below is based on the following extension of Fubini' s theorem.

Theorem 1. *If a real function $f(x)$ is integrable on R_n and is equal to zero almost everywhere on F, then*

$$\int f\, dm = \int_0^\infty \int_{E_t} f\, d\nu\, dt, \tag{1}$$

where m is Lebesgue measure in R_n, and ν is the $(n-1)$-dimensional Hausdorff measure (([1], p. 111, or ([2], p. 92), defined on subsets of the subspace $E_t \subset R_n$.

The proof of this theorem is carried out by the standard method of the theory of integration, starting from the representation, given in our work ([3]), of the Lebesgue measure of a measurable set $M \subset R_n$ with measure $m(M \cap F) = 0$:

$$mM = \int_0^\infty \nu(E_t \cap M)\, dt. \tag{2}$$

Let us note that the usual formulation of Fubini' s theorem for Euclidean $(k+l)$-dimensional space is obtained from Theorem 1 by induction on the number k (as the set F one then takes a $(k-1+l)$-dimensional coordinate plane).

The assertions formulated below concern mappings of plane domains; however, apparently, with corresponding modifications, they can be extended to domains of the space R_n of higher dimension.

Using relation (2), it is not difficult to prove that, for almost all $t \in [0, \infty)$, every simple arc $\Gamma \subset E_t$ is rectifiable.

Theorem 2. *Let G be a domain and F a closed set in the z-plane R_2. If a real-valued function $f(x, y)$ is continuously differentiable in the domain G, then for almost all $t \in (t_1, t_2)$, $t_1 = \inf_{z \in G} r(z)$, $t_2 =$*

$$= \sup_{z \in G} r(z)$$

the composite function $f[z(t, s)] \equiv f[x(t, s), y(t, s)]$ is absolutely continuous inside $E_t \cap G$, i.e., is absolutely continuous on every simple arc $\Gamma \subset E_t \cap G$ as a function of the arc length s of the curve Γ.

As is known, the derivative of a continuously differentiable function $f(x, y)$ in the direction of the tangent s to $E_t \cap G$ (at points where such a tangent exists) is equal to $|g_f| \cos(g_f, s)$, where g_f denotes the gradient of the function f.

If the real and imaginary parts of the mapping $T(z) = u(z) + iv(z)$ of the domain G are continuously differentiable, then, using Theorems 1 and 2, one obtains the equality:

$$\int_{t_1}^{t_2} l[T(E_t \cap G)]\, dt = \iint_{G \setminus F} \left\{|g_u|^2 \cos^2 |g_u, s| + |g_v|^2 \cos^2 (g_v, s)\right\}^{1/2} dx\, dy, \quad (3)$$

where l is the sum of the lengths of the images of the arcs comprising the intersection $E_t \cap G$, and g_u and g_v are the gradients of the functions u and v, respectively.

From equality (3) follows the relation

$$\int_{t_1}^{t_2} l[T(E_t \cap G)]\, dt \leq \iint_{G \setminus F} (|u_x| + |u_y| + |v_x| + |v_y|)\, dx\, dy, \quad (4)$$

which, by means of a suitable smoothing process (see, for example, ([4]), p. 218), extends to continuous mappings with generalized first partial derivatives in G ([4]), p. 338). The class of such mappings is denoted by $D(G)$.

Inequality (4) shows that if the first generalized derivatives of the mapping $T(z) \in D(G)$ are summable on $G \setminus F$, then the quantity $l[T(E_t \cap G)]$ is finite for almost all $t \in (t_1, t_2)$. A consequence of (3) is the inequality

$$\int_{t_1}^{t_2} l[T(E_t \cap G)]\, dt \leq \left\{ m(G \setminus F) \cdot \iint_{G \setminus F} (|g_u|^2 + |g_v|^2)\, dx\, dy \right\}^{1/2}$$

(cf. ([5]), p. 178 or ([6])), which also remains valid for mappings $T \in D(G)$ and is nontrivial if the factors on the right are both finite.

Theorem 3. Let the Dirichlet integral

$$I_G(T) = \iint_G (|T_x|^2 + |T_y|^2)\, dx\, dy$$

of the mapping $T(z) \in D(G)$ be finite, i.e., let T belong to the class $BL(G)$ (see ([7])). Then

$$\int_{t_1}^{t_2} \frac{l^2[T(E_t \cap G)]}{l(E_t \cap G)}\, dt \leq \iint_{G \setminus F} (u_x^2 + v_x^2 + u_y^2 + v_y^2)\, dx\, dy, \tag{5}$$

where the integrand in the left-hand side should be regarded as equal to zero if the length $l(E_t \cap G) = \infty$ or $l[T(E_t \cap G)] = 0$.

It is easy to show, however, that relation (5) also has meaning for mappings $T(z) \in D(G)$ whose derivatives are summable on the set $G \setminus F$.

Let us note that if the set F consists of a single point $a \neq \infty$, then from (5) one immediately obtains the well-known inequality

$$\int_{t_1}^{t_2} l^2[T(E_t \cap G)]\, \frac{dt}{t} \leq 2\pi I_G(T),$$

which was used as the basis for the investigations in ([5,7]).

Next, let

$$T(w) = x(u, v) + iy(u, v)$$

be an (interior) quasiconformal (([8]), p. 24) mapping of the domain (or open set) Δ

in the z-plane, and $n(z)$ is the number of roots of the equation $h(w) = z$ lying in Δ. It can be shown that the function $n(z)$ is measurable. Put $p(t) = \int_{E_t} n(z)\, ds$, where ds is the element of arc length and $z \in E_t$.

The following proposition is a generalization of the well-known principle of length and area (see ([9], [10])).

Theorem 4. Suppose that the function $T(w)$ carries out an (interior) Q-quasiconformal mapping of the open set Δ with area S_Δ, and $G = T(\Delta)$. Then

$$\int_{t_1}^{t_2} l^2[T^{-1}(E_t \cap G)]\, \frac{dt}{p(t)} \leq Q S_\Delta, \tag{6}$$

where $l[T^{-1}(E_t \cap G)]$ is the total length of the curves in Δ constituting the preimage of the set $E_t \cap G$ (the expression under the integral on the left is regarded as equal to zero if $p(t) = \infty$ or $l[T^{-1}(E_t \cap G)] = 0$).

Using Theorem 3, the following boundary property is proved.

Theorem 5. Let $T(z) \in BL(G)$ be a topological mapping of the domain G (with boundary ∂G) onto a domain $\Delta = T(G)$ with rectifiable boundary. If $F \subset \partial G$ is a closed set and

$$\lim_{t \to 0} \frac{l(E_t \cap G)}{t} < \infty,$$

then the cluster set* $C_F(T)$ has length zero.

It would be interesting to compare this result with Theorems 3 and 4 of the paper ([11]), which are close in character to the last theorem.

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* The cluster set $C_N(T)$ is the totality of all limit points of all possible sequences of the form $\{T(z_n)\}$, where $z_n \in G$, $\lim_n z_n = z_0 \in N \subset \overline{G}$.

Note: Figure translations are in progress. See original paper for figures.