FINE AND COARSE CONVERGENCE OF INTERPOLATION AND QUADRATURE PROCESSES

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Abstract

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FINE AND COARSE CONVERGENCE OF INTERPOLATION AND QUADRATURE PROCESSES

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Let $A(a_{in})$ $(i = 1, 2, \dots, n;\ n = 1, 2)$ be an infinite triangular numerical matrix satisfying the condition $-1 \le a_{in} \le 1$; let $f(x)$ be a function continuous on $[-1, 1]$, and let $\beta > 0$ be an arbitrary number. Denote by $L_n(f, x, A)$ the Lagrange interpolation polynomial of the function $f(x)$ with interpolation nodes a_{in} $(i = 1, 2, \dots, n)$, and by $M(\beta)$ the class of matrices A for which, for every positive $\varepsilon < \beta$,

$$\overline{\lim_{n\to\infty}} \frac{\|L_n(f, x, A)\|}{n^{\beta+\varepsilon}} < c_1(\varepsilon), \qquad \overline{\lim_{n\to\infty}} \frac{\|L_n(f, x, A)\|}{n^{\beta-\varepsilon}} > c_2(\varepsilon),$$

where $c_1(\varepsilon)$ and $c_2(\varepsilon)$ are certain nonnegative constants,

$$\|L_n(f, x, A)\| = \max_{x\in[-1,1]} \sup_{|f(t)|\le 1} |L_n(f, x, A)|.$$

Considering the case when $0 < \beta < 1$ and the classes $\operatorname{Lip}\alpha$ of functions satisfying the Lipschitz condition of order α, P. Erdős and P. Turán (^{1}) showed that, for $\beta/(\beta + 2) < \alpha < \beta$, there exists an interpolation process with a matrix $A_1 \in M(\beta)$ that is uniformly convergent on the whole class $\operatorname{Lip}\alpha$, and also an interpolation process with a matrix $A_2 \in M(\beta)$ and a function $f_1(x) \in \operatorname{Lip}\alpha$ such that the sequence

$$\max_{x\in[-1,1]} |L_n(f_1, x, A_2)|$$

diverges without bound as $n \to \infty$. P. Erdős and P. Turán called the interpolation theory corresponding to this case "fine," in contrast to the "coarse" theory, where convergence of the process depends only on the growth rate of the sequence of its norms.

In (^{1}) an essential role was played by the restriction $0 < \beta < 1$, naturally connected with the Lipschitz condition. In this note arbitrary positive values of β are considered, and a generalization is given of the aforementioned results on fine and coarse interpolation for classes of functions defined by the behavior of the modulus of smoothness of the corresponding order; analogous results relating to the theory of mechanical quadratures are also presented.

Let, for a function $f(x)$ continuous on the interval $[-1, 1]$,

$$\omega_r(f;t) = \sup_{x,\ x+rh\in[-1,1],\ |h|\le t} \left| \sum_{\nu=0}^{r} (-1)^{r-\nu} \binom{r}{\nu} f(x+\nu h) \right|$$

be the modulus of smoothness of this function of order r.

Denote by $H_\alpha^{(r)}$ the Banach space of functions $f(x)$ satisfying the condition $\omega_r(f;t) \le kt^\alpha$, where k is some constant and $\alpha > 0$, with norm

$$\|f\|_{H_\alpha^{(r)}} = \max_{x\in[-1,1]} |f(x)| + \sup_t \frac{\omega_r(f,t)}{t^\alpha}.$$

For $\beta < r$ the following assertions hold.

Theorem 1. *Whatever the matrix $A \in M(\beta)$ and positive-*

value $\alpha < \beta/(\beta+2)$, there exists a function $f_1(x) \in H_\alpha^{(r)}$ for which the sequence

$$\max_{x\in[-1,1]} |L_n(f_1, x, A)|$$

is unbounded.

This theorem is a consequence of the fact that, for any matrix $A \in M(\beta)$, from the sequence

$$\|L_n(f, x, A)\|_{H_\alpha^{(r)}} = \max_{x\in[-1,1]} \sup_{\|f\|_{H_\alpha^{(r)}}\le 1} |L_n(f, x, A)|$$

one can extract a subsequence satisfying the relation

$$\|L_{n_i}(f, x, A)\|_{H_\alpha^{(r)}} > c_3 n_i^{[\beta-(\beta+2)]/(1+\alpha)}.$$

Theorem 2. There exists a matrix $A_1 \in M(\beta)$ such that, for any $\alpha > \beta/(\beta+2)$, the sequence $L_n(f, x, A_1)$ converges uniformly to $f(x)$ for all $f(x) \in H_\alpha^{(r)}$.

Theorem 3. If $\alpha < \beta$, there exist a matrix $A_2 \in M(\beta)$ and a function $f_2(x) \in H_\alpha^{(r)}$ such that the sequence

$$\max_{x\in[-1,1]} |L_n(f_2, x, A_2)|$$

is unbounded.

In this case, as for Theorem 1, one can give an equivalent formulation: there exists a matrix $A_2 \in M(\beta)$ such that

$$\|L_{n_i}(f, x, A_2)\|_{H_\alpha^{(r)}} > c_4 n_i^{(\beta-\alpha)/2}.$$

Theorem 4. For any matrix $A \in M(\beta)$ and $f(x) \in H_\alpha^{(r)}$, the sequence $L_n(f, x, A)$, for $\alpha > \beta$, converges uniformly to the function $f(x)$.

Results of an analogous character also hold for mechanical quadrature formulas.

Let the quadrature process be constructed on the basis of the interpolation polynomials $L_n(f, x, A)$. Denote by $A(\beta)$ the class of matrices A for which, for every positive $\varepsilon < \beta$,

$$\overline{\lim_{n\to\infty}} \frac{\|Q_n(f, A)\|}{n^{\beta+\varepsilon}} < c_5(\varepsilon), \qquad \underline{\lim_{n\to\infty}} \frac{\|Q_n(f, A)\|}{n^{\beta-\varepsilon}} > c_6(\varepsilon),$$

where $c_5(\varepsilon)$ and $c_6(\varepsilon)$ are nonnegative constants,

$$\|Q_n(f, A)\| = \sup_{|f(t)|\le 1} \left| \int_{-1}^{1} L_n(f, x, A)\, dx \right|.$$

In this case, for $r > \beta$, the following assertions hold.

Theorem 5. For any matrix $A \in A(\beta)$ and function $f(x) \in H_\alpha^{(r)}$, for $\alpha > \beta$ the sequence $Q_n(f, A)$ converges to

$$\int_{-1}^{1} f(x)\, dx.$$

Theorem 6. There exists a matrix $A_3 \in A(\beta)$ such that, for $\alpha > \beta/(\beta + 4)$,

$$\lim_{n\to\infty} Q_n(f, A) = \int_{-1}^{1} f(x)\, dx$$

for all functions $f(x)$ from $H_\alpha^{(r)}$.

Theorem 7. If $\alpha < \beta$, then there exist a matrix $A_4 \in A(\beta)$ and a function $f_4(x) \in H_\alpha^{(r)}$ such that the sequence $Q_n(f_4, A_4)$ is unbounded.

Theorem 8. Whatever the matrix $A \in A(\beta)$, whose elements satisfy the condition

$$|a_{jn} - a_{in}| > c_7(\varepsilon) n^{-\beta-4-\varepsilon}, \qquad i \neq j$$

(ε is an arbitrary positive number), for any $\alpha < \beta/(\beta+4)$ there exists a function $f_3(x) \in H_\alpha^{(r)}$ such that the sequence $Q_n(f_3, A)$ is unbounded.

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References

1. P. Erdös, P. Turán, *Acta Math. Acad. Sci. Hung.*, **6**, No. 1-2 (1955).

Note: Figure translations are in progress. See original paper for figures.