# CYBERNETICS AND CONTROL THEORY

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## Abstract

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# ON THE PRINCIPLE OF CONSTRUCTING A SEARCH-FREE ADAPTIVE AUTOMATIC CONTROL SYSTEM WITH VARIABLE STRUCTURE

Problems of constructing high-quality control systems have been solved in various specific formulations. In ($^{1,2}$) the use, for this purpose, of reference models in estimating the parameters of the plant at a rate commensurate with the duration of the transient processes made it possible substantially to improve the quality indices of the latter. The known successes along this path can be achieved by applying the principle of adaptation in automatic control systems with variable structure ($^{3,4}$). Let us consider the problem of automatic control of a plant whose gain coefficient varies over wide limits, while the rate of change is small in comparison with the rate at which the transient process proceeds.

Let, in the $n$-dimensional phase space of coordinates $x_1, \dots, x_n$, the motion of the dynamical system be described by the equation

$$dx/dt = \mathbf{f}(\mathbf{x}), \tag{1}$$

where $\mathbf{x} = \{x_1, \dots, x_n\}$, $\mathbf{f} = \{f_1, \dots, f_n\}$ are $n$-dimensional vectors; $f_i = x_{i+1}$, $i = 1, 2, \dots, n-1$,

$$f_n = -\sum_{i=1}^{n} a_i x_i - \psi(\mathbf{x}) b x_1;$$

$a_i$ are constant coefficients ($i = 1, \dots, n$); $b$ is the gain coefficient of the plant, varying within the limits $0 < b_{\min} \leq b \leq b_{\max}$; $\psi(\mathbf{x})$ is a controllable parameter of the regulator, which may take one of two values $\omega_1 > 0$ or $\omega_2 < 0$.

For any fixed value $b \in [b_{\min}, b_{\max}]$, the logical law of variation of the parameter $\psi$

$$\psi = \begin{cases} \omega_1 & \text{for } gx_1 > 0, \\ \omega_2 & \text{for } gx_1 < 0, \end{cases} \tag{2}$$

where $g = \mathbf{c} \cdot \mathbf{x}$ is a certain switching function, and $\mathbf{c} = \{c_1, \dots, c_n\}$ is a fixed vector, can be chosen in such a way (5) that, after reaching the hyperplane $S$ defined by the equation

$$g = \mathbf{c} \cdot \mathbf{x} = 0, \tag{3}$$

the representative point will move in a sliding regime along phase trajectories belonging to $S$ (6). For this it is sufficient that the relations

$$c_{i-1} - a_i = (c_{n-1} - a_n)c_i, \quad i = 2, 3, \dots, n; \quad c_n = 1; \tag{4}$$

$$\omega_1 b > -c_1(c_{n-1} - a_n) - a_1,$$

$$\omega_2 b < -c_1(c_{n-1} - a_n) - a_1. \tag{5}$$

be satisfied.

We shall show that the dynamic properties of the system (1), (2) can be improved if, in contrast to (4,5), the vector $\mathbf{c}$ is chosen not fixed, but varying according to a certain law as a function of the value of the coeff-

gain coefficient $b$ in the transient process. Let us denote in (4) $c_{n-1} - a_n = k$, where $k \in (-\infty, +\infty)$. Then the vector-function $c(k)$ is single-valued and continuous on $(-\infty, +\infty)$.

Substitute $c_1(k)$ into (5):

$$\omega_1 b > F(k), \qquad \omega_2 b < F(k), \tag{6}$$

where

$$F(k) = (-1)(k^n + k^{n-1}a_n + \dots + a_1)$$

is a polynomial differing from the characteristic polynomial of the open-loop system only in sign.

Divide the interval $[b_{\min}, b_{\max}]$ into $l$ parts by division points

$$b_0 = b_{\min}, \; b_1, \dots, b_i, \dots, b_j, \dots, b_l = b_{\max}$$

so that $b_i < b_j$ for $0 \le i < j \le l$.

Denote

$$\Omega_{1j} = \omega_1 b_j, \qquad \Omega_{2j} = \omega_2 b_j. \tag{7}$$

Let $\{F_j(k)\}$ be the set of all values of $F(k)$ satisfying the conditions

$$\Omega_{2j} \le F(k) \le \Omega_{1j}, \qquad j = 0, 1, \dots, l. \tag{8}$$

To this set we assign the set of arguments $K_j$ such that $F(k) \in \{F_j(k)\}$ for $k \in K_j$.

Obviously,

$$\{F_i(k)\} \subset \{F_j(k)\} \quad \text{for } i < j \tag{9}$$

and, consequently, $K_i \subset K_j$.

For each $j = 0, 1, \dots, l-1$, choose such a $k_j \in K_j$ that the motion of the representative point along the hyperplane $S_{k_j}$ (the hyperplane $S$ with normal vector $c(k_j)$) after the onset of the sliding mode will be the best in some sense, for example, in the sense of the highest degree of stability (7), over all $k \in K_j$. Then one may assert that the motion of the representative point in the sliding mode along the hyperplane $S_{k_j}$ will be no worse than in the hyperplane $S_{k_i}$ for $i < j$. On this basis we modify the logical control law (2):

$$\psi = \begin{cases} \omega_1 & \text{for } gx_1 > 0, \\ \omega_2 & \text{for } gx_1 < 0; \end{cases} \tag{10a}$$

$$g = \begin{cases} c(k_j)x & \text{for } b_j \le b < b_{j+1}, \quad j = 0, 1, \dots, l-2, \\ c(k_{l-1})x & \text{for } b_{l-1} \le b \le b_l. \end{cases} \tag{10b}$$

The proposed control law makes it possible to solve the stated problem, since for sufficiently small

$$\Delta b = \max_j \Delta b_j \qquad (\Delta b = b_j - b_{j-1})$$

the transient process in system (1), (10a), (10b) will be the best, or arbitrarily close to the best, in the sense of the selected criterion. To implement this law, it is necessary to know the value of $b$ in the transient process. Let us consider one of the possible ways of estimating $b$.

Introduce the function

$$\sigma = ex \tag{11}$$

where $e = \{e_1, \dots, e_n\}$, $e_n = 1$.

The equation $\sigma = 0$ defines in phase space a certain hyperplane $P$ with normal vector $e$. For a fixed value of $\psi(x)$, the function $\sigma$ is continuously differentiable with respect to $t$:

$$\dot{\sigma} = e\frac{dx}{dt} = -\psi b x_1 + \sum_{i=1}^{n} (e_{i-1} - a_i) x_i, \qquad e_0 = 0. \tag{12}$$

We require that at the instant $t_p$ such that the radius vector of the representative point

$$x(t_p) = x_p = \{x_{1p}, \dots, x_{np}\} \in P,$$

the value $\dot{\sigma}$ not depend on the coordinates $x_{ip}$, $i = 2, 3, \dots, n$, for all $x_p \in P$.

Taking (11) and (12) into account, one can write

$$\dot{\sigma}(t_p) = -\psi b x_{1p} + \sum_{i=1}^{i-1} \left[(e_{i-1} - a_i) - (e_{n-1} - a_n) e_i\right] x_{ip} \tag{13}$$

The required condition is satisfied if

$$\sum_{i=1}^{n-1} \left[(e_{i-1} - a_i) - (e_{n-1} - a_i) e_i\right] x_{ip} = 0, \qquad e_0 = 0 \tag{14}$$

for all $x_p \in P$.

Instead of (14), one may consider the equivalent algebraic system

$$e_{i-1} - a_i = k e_i, \qquad i = 1, \dots, n, \qquad e_0 = 0, \qquad e_n = 1, \tag{15}$$

where $k = e_{n-1} - a_n$.

The vector $\mathbf{e}$, which is a solution of (15), is normal to an integral hyperplane of system (1) for $\psi(\mathbf{x}) = 0$, and the number of these hyperplanes is equal to the number of distinct real roots of the characteristic polynomial of the open-loop system. It is important to note that a real solution of system (15) exists if and only if $k$ is equal to one of the real roots of the polynomial $F(k)$ (see (6)).

Let us consider two cases.

**Case 1.** The polynomial $F(k)$ has real roots. Let $k^*$ be one of the real roots, and let the vector $\mathbf{e}_0 = \{e_{10}, \dots, e_{n0}\}$, $e_{n0} = 1$, be a solution of (15) for $k = k^*$ and be normal to the hyperplane $P^0$ passing through the origin. Then

$$\dot{\sigma}(t_p) = \psi b x_{1p}. \tag{16}$$

Let us give the argument $t$ an increment $\Delta t$. At the time $t_q = t_p + \Delta t$, the representative point will be in the position $\mathbf{x}(t_q) = \mathbf{x}_q = \{x_{1q}, \dots, x_{nq}\}$, and the function $\sigma$ will receive the increment

$$\Delta\sigma = \mathbf{e}_0 \cdot \mathbf{x}_q - \mathbf{e}_0 \cdot \mathbf{x}_p = \mathbf{e}_0 \cdot \mathbf{x}_q. \tag{17}$$

On the other hand, for sufficiently small $\Delta t$,

$$\Delta\sigma \simeq \psi b x_{1p} \Delta t, \tag{18}$$

and $x_{1p} \simeq x_{1q}$. Subtracting (18) from (17), we obtain

$$\mathbf{e}_0 \mathbf{x}_q - \psi b \Delta t x_{1q} \simeq 0. \tag{19}$$

When (19) is strictly satisfied, there is an equation of a pencil of hyperplanes $\{Q\}$, and for $\Delta t = 0$ it defines the hyperplane $P^0$. For chosen $\mathbf{e}_0$, $\psi$, and fixed $\Delta t_0$, the hyperplane $Q \in \{Q\}$ is the closer to the hyperplane $P^0$ the smaller the value $b \in [b_{\min}, b_{\max}]$, i.e., for $b = b_0$. Denote this hyperplane by $Q^0$. Its equation is

$$\mathbf{e}_0 \mathbf{x} + \lambda x_1 = 0, \tag{20}$$

where $\lambda = \psi b_0 \Delta t_0$.

Suppose now that the transition from the point $\mathbf{x}_p \in P^0$ ($x_{1p} \neq 0$) to the point $\mathbf{x}_q \in Q^0$ (or, in view of the approximate nature of (18), to a sufficiently small neighborhood of it) is carried out for different values $b \in [b_{\min}, b_{\max}]$. Obviously, in this case

$$\lambda = \psi b \Delta t, \tag{21}$$

where $\Delta t$ is the transition time.

Then from (20) and (21),

$$\Delta t(b) = b_0 \Delta t_0 / b. \tag{22}$$

Thus, the time required for the representative point to traverse the distance between two fixed hyperplanes $P^0$ and $Q^0$ is a monotone function of the gain coefficient $b$ and does not depend on the initial conditions.

In view of the existence of the inverse function $b(\Delta t)$, the problem of estimating the quantity $b$ can be reduced to that of estimating the quantity $\Delta t$.

**Case 2.** The polynomial $F(k)$ has no real roots. Instead of the system (15), let us consider the "shortened" system

$$e_{i-1} - a_i = ke_i, \quad i = 2, 3, \dots, n, \quad e_n = 1. \tag{23}$$

This system has a unique solution for any $k$. Let the vector $\mathbf{e}_0 = \{e_{10}, \dots, e_{n0}\}$, $e_{n0} = 1$, normal to a certain hyperplane $P_0$, be the solution of (23) for $k = k^{**}$. Then

$$\dot{\sigma} = -\psi b x_{1p} + F(k^{**}) x_{1p}. \tag{24}$$

Choose the quantity $\psi(x)$ so that, for all $b \in [b_{\min}, b_{\max}]$, the function $\dot{\sigma}$ does not vanish for all $x_p \in P^0$ $(x_{1p} \neq 0)$. For this it is sufficient that

$$|\psi b_0| > |F(k^{**})|. \tag{25}$$

Then the arguments and conclusions obtained for case 1 are also valid for case 2; in this case

$$\Delta t = \frac{\psi b_0 - F(k^{**})}{\psi b - F(k^{**})} \Delta t_0. \tag{26}$$

Thus we have reduced the problem of estimating the quantity $b$ in the transient process to the problem of estimating the transition time $\Delta t$ of the representative point from one hyperplane fixed in the phase space to another. This problem is of a technical character and is solved by means of an adaptive control device which measures the quantity $\Delta t$ and, in accordance with its value, forms the switching function $g$ according to (10b). In other words, the adaptive control device plays the role of a functional converter realizing the function

$$g = \begin{cases} c(k_j) \cdot x, & \text{for } \Delta t_{j+1} < \Delta t \leq \Delta t_j, \quad j = 0, 1, \dots, l-2, \\ c(k_{l-1}) \cdot x, & \text{for } \Delta t_l \leq \Delta t \leq \Delta t_{l-1}, \end{cases} \tag{27}$$

where $\Delta t_j = \Delta t(b_j)$.

The process of estimating the quantity $b$ and establishing the required value of the switching function takes place in system (1), (10a), (27) during the transient process for fixed $\psi$. Then, on the basis of the estimate, the switching function

$g$ (27) is established in the system. It is fixed during the transient process and until the representative point is held in some prescribed neighborhood of the origin of coordinates. When it leaves this neighborhood, the entire cycle is repeated from the beginning.

Thus it has been shown that in the automatic control system (1), (2) one can achieve an improvement in quality in the sense of the selected criterion (degree of stability) by introducing an adaptive control device that realizes the logical function (27). The constructed control system belongs to the class of non-search adaptive systems.

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*Note: Figure translations are in progress. See original paper for figures.*