INVERTIBILITY IN TOPOLOGICAL RINGS

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Abstract

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INVERTIBILITY IN TOPOLOGICAL RINGS

It is known (see, for example, ([1])) that any topological ring R can be embedded as an everywhere dense subring in a complete ring $\hat{R}$, i.e., a ring in which every Cauchy filter has a limit ($\hat{R}$ is determined up to isomorphism and is called the completion of the ring R).

In the present paper it is proved that, in order that the completion $\hat{R}$ of a topological ring R contain an identity and that every element of R be invertible in $\hat{R}$, it is necessary and sufficient that R contain no closed one-sided ideals* and no generalized zero divisors (for the definition of a generalized zero divisor see below). From this, as a consequence, it follows that a complete ring containing no closed one-sided ideals and no generalized zero divisors is a field. It is proved that a locally bicompact ring R with identity, containing no closed one-sided ideals, is a field.

Lemma 1. *In order that a topological ring R with nonzero multiplication contain no closed left (right) ideals, it is necessary and sufficient that for any elements $0 \neq a, b \in R$ and any neighborhood of zero V** there exist an element $x \in R$ such that $xa + b \in V$ $(ax + b \in V)$.*

Sufficiency. Let I be some left ideal. We shall show that $[I]_R = R$***. Indeed, if b is some element of R and $0 \neq a \in I$, then for any neighborhood of zero V there exists an element $x \in R$ such that $xa + b \in V$. Then $xa \in -b + V$. Consequently, every neighborhood of the element $-b$ has a nonempty intersection with I. Hence $-b \in [I]_R$. From the arbitrariness of b it follows that $[I]_R = R$.

Necessity. Let $a \neq 0$ and b be some elements of R, and let V be any neighborhood of zero. Consider the left ideal Ra. If $Ra = 0$, then $I = \{c \mid Rc = 0\}$ is a closed ideal. Since $a \in I$, it follows that $I = R$. Then $R^2 = 0$. But this is impossible, since R is a ring with nonzero multiplication. Hence $Ra \neq 0$. Since $[Ra]_R = R$, we have $-b \in [Ra]_R$. Consequently, $-b - V \cap R \cdot a = \emptyset$, and hence $b + xa \in V$ for some $x \in R$.

Recall that an element a of a topological ring R is called a **left (right) generalized zero divisor** if there exists a neighborhood of zero V such that for every neighborhood of zero U one can find an element $x \notin V$ for which $ax \in U$

$(xa \in U)$. An element a is called a **generalized zero divisor** if a is a left and right generalized zero divisor.

Lemma 2. *If a is not a left (right) generalized zero divisor, then a^2 is also not a left (right) generalized zero divisor.*

Lemma 3. *If the completion $\hat{R}$ of a topological ring R contains an identity e and every element of R is left (right) invertible in $\hat{R}$, then R contains no closed left (right) ideals.*

* By an ideal we shall mean a proper ideal, i.e., a nonzero ideal distinct from the whole ring.

** By a neighborhood of a point x we shall mean any set containing an open set that contains the point x.

*** If M is some set in a topological ring R, then $[M]_R$ denotes the closure of M in R.

Lemma 4. If the completion $\hat{R}$ of a topological ring R contains an identity e and an element $a \in R$ is left (right) invertible in $\hat{R}$, then a is not a left (right) generalized zero divisor in R.

Lemma 5. If a topological ring R contains no nonzero closed one-sided ideals and an element $a \in R$ is not a right (left) generalized zero divisor, then the completion $\hat{R}$ contains an identity e, and a is left (right) invertible in $\hat{R}$.

Proof. If $\{V_\alpha\}$ is some base of neighborhoods of zero, then, since a is not a right generalized zero divisor, there exists a family $\{U_\alpha\}$ of neighborhoods of zero in R such that from $xa^2 \in U_\alpha$ it follows that $x \in V_\alpha$. There exists a base of neighborhoods of zero $\{W_\alpha\}$ in R such that $W_\alpha - W_\alpha \subseteq U_\alpha$. Since $a^2 \neq 0$, R cannot be a ring with zero multiplication. By Lemma 1, for every neighborhood W_α there exists $x_\alpha \in R$ such that $x_\alpha a^2 - a \in W_\alpha$.

For each α define the set

$$\mathfrak{R}_\alpha = \{x_\gamma \mid x_\gamma a^2 - a \in W_\alpha\}.$$

Since

$$\mathfrak{R}_\alpha \cap \mathfrak{R}_\beta = \{x_\gamma \mid x_\gamma a^2 - a \in W_\alpha \cap W_\beta\} \supseteq \mathfrak{R}_\rho,$$

where ρ is such that $W_\rho \subseteq W_\alpha \cap W_\beta$, the family of sets $\mathfrak{R}_\alpha$ is a base of some filter F.

We prove that the filter thus obtained is a Cauchy filter. For this it is enough to prove that $\mathfrak{R}_\alpha - x_\alpha \subseteq V_\alpha$. If $x_\gamma \in \mathfrak{R}_\alpha$, then $x_\gamma a^2 - a \in W_\alpha$ and $x_\alpha a^2 - a \in W_\alpha$. Hence

$$(x_\gamma - x_\alpha)a^2 \in W_\alpha - W_\alpha \subseteq U_\alpha.$$

From the definition of U_α it follows that $x_\gamma - x_\alpha \in V_\alpha$. Consequently, $\mathfrak{R}_\alpha - x_\alpha \subseteq V_\alpha$.

Let $\hat{x}$ be the limit of the filter F in $\hat{R}$. Put $e = \hat{x}a$ and show that e is an identity in $\hat{R}$.

Since $\mathfrak{R}_\alpha a^2 - a \subseteq W_\alpha$ and $\hat{x}$ is the limit of the filter F, we have $\hat{x}a^2 - a = 0$, i.e. $\hat{x}a^2 = a$, and therefore $ea = a$. Since R contains no closed right ideals, aR is everywhere dense in R, and hence aR is everywhere dense in $\hat{R}$. Now let b be some element of $\hat{R}$, and suppose that $eb-b \neq 0$. There exists a neighborhood of zero V in $\hat{R}$ such that $eb-b \notin V$, and a neighborhood of zero U in $\hat{R}$ such that $U + eU \subseteq V$. Since aR is everywhere dense in $\hat{R}$, it follows that there exists an element $c \in R$ for which $ac - b \in U$. Then

$$eb - b = e(b - ac) + eac - b \in e \cdot U + U \subseteq V.$$

We have obtained a contradiction to the assumption. Consequently, $eb - b = 0$, i.e. e is a left identity in $\hat{R}$.

We now show that e is also a right identity in $\hat{R}$. Suppose the contrary, i.e. that $be - b \neq 0$ for some $b \in \hat{R}$. There exists a neighborhood of zero W in $\hat{R}$ such that

$$(W + be - b) \cap W = \emptyset.$$

Since R is everywhere dense in $\hat{R}$, for any neighborhood of zero W'_α from the base of neighborhoods of zero $\{W'_\alpha\}$ in $\hat{R}$,

$$R \cap (be - b + (W \cap W'_\alpha)) \neq \emptyset.$$

Let

$$y_\alpha \in R \cap (be - b + (W \cap W'_\alpha)),$$

and let M be the family of all such y_α. Then

$$M \cap W \subseteq (be - b + W) \cap W = \emptyset.$$

If V is an arbitrary neighborhood of zero in $\hat{R}$, then there exists a neighborhood of zero W'_γ such that $W'_\gamma \cdot a \subseteq V$. Then

$$y_\gamma \cdot a \in R \cap ((be - b + W'_\gamma)a) = R \cap (bea - ba + W'_\gamma a) = R \cap (W'_\gamma a) \subseteq R \cap V.$$

From the fact that $y_\gamma \notin W$ and the arbitrariness of V it follows that a is a right generalized zero divisor in R. We have obtained a contradiction. Consequently, $be = b$ for all $b \in \hat{R}$, and hence e is a two-sided identity in $\hat{R}$. Then $\hat{x}$ is a left inverse for a.

Theorem 1. In order that the completion $\hat{R}$ of a topological ring R contain an identity e and that every element of R be invertible in $\hat{R}$, it is necessary and sufficient that R contain no closed one-sided ideals and no generalized zero divisors.

Proof. The necessity follows easily from Lemmas 3 and 4,

Sufficiency. Let a be an arbitrary element of R. Since R contains no generalized zero divisors, a is not either a right or a left generalized zero divisor. Suppose, for definiteness, that a is not a right generalized zero divisor. Then, by Lemma 5, $\hat{R}$ has an identity and a is left-invertible in $\hat{R}$. By Lemma 4, a is not a left generalized zero divisor. Applying Lemma 5 once more, we obtain that a is right-invertible in $\hat{R}$. Hence a is invertible in $\hat{R}$.

Corollary. *A complete topological ring R containing no generalized zero divisors and no closed one-sided ideals is a field.*

Indeed, the completion $\hat{R}$ has an identity and every element of R is invertible in $\hat{R}$. Since $\hat{R}$ is a complete topological ring, $R = \hat{R}$. Consequently, R itself has an identity and every element of R is invertible in R.

As usual, a set M of a topological ring R will be called **topologically nilpotent** if for every neighborhood V of zero there exists a number n such that

$$M^k = \{a_1 a_2 \dots a_k \mid a_i \in M\} \subseteq V$$

for all $k \geq n$.

Theorem 2. *If a topological ring R with identity e has a topologically nilpotent neighborhood of zero V and contains no closed one-sided ideals, then every element of R is invertible in the completion $\hat{R}$.*

Proof. Let n be such a number that

$$V^k + V^{k+1} \subseteq V$$

for all $k \geq n$. For any element $a \in R$, by Lemma 1, there exists an element $x \in R$ such that

$$ax + e \in V.$$

Then

$$(ax + e)^n = ax \sum_{i=1} C_n^i (ax)^{i-1} + e \in V^n.$$

Consequently, there exists an element

$$y = x \sum_i C_n^i (ax)^{i-1},$$

such that

$$ay + e \in V^n.$$

For any number p, put

$$s_p = \sum_{k=0}^{p} y(ay + e)^k.$$

We shall show that the sequence of elements $s_1, s_2, \dots$ is a Cauchy sequence, i.e. that for every neighborhood of zero U there exists a number m such that

$$s_p - s_q \in U$$

for all $p, q \geq m$.

Let W be a neighborhood of zero in R such that

$$y \cdot W \subseteq U,$$

and let m be such a number that

$$V^k \subseteq W$$

for $k \geq m$. Then

$$s_p - s_q = y \sum_{k=q+1}^{p} (ay+e)^k = y(ay+e)^q \sum_{k=1}^{p-q} (ay+e)^k.$$

We shall show that

$$\sum_{k=1}^{r} (ay+e)^k \in V$$

for every r. Indeed, for $r = 1$ we have

$$(ay+e) \in V^n \subseteq V.$$

Suppose that

$$\sum_{k=1}^{r} (ay+e)^k \in V.$$

Then

$$\sum_{k=1}^{r+1} (ay+e)^k = (ay+e) + (ay+e) \sum_{k=1}^{r} (ay+e)^k \in V^n + V^n \cdot V = V^n + V^{n+1} \subseteq V.$$

Consequently,

$$s_p - s_q \in y(ay+e)^q \cdot V \subseteq y \cdot V^{q+1} \subseteq y \cdot W \subseteq U$$

for $p, q \geq m$. From the arbitrariness of U it follows that the sequence $s_1, s_2, \ldots$ is a Cauchy sequence.

There exists an element $d \in \hat{R}$ such that

$$d = \lim_p s_p$$

(i.e. every neighborhood of the element d contains all elements s_p, starting from some one). But

$$\begin{aligned} ad &= ad - as_p + as_p \\ &= a(d - s_p) + as_p + e - e \\ &= a(d - s_p) + a\sum_{k=0}^{p} y(ay+e)^k + e - e \\ &= a(d - s_p) + \left(\sum_{k=0}^{p} ay(ay+e)^k + e\right) - e \\ &= a(d - s_p) + (ay+e)^{p+1} - e \\ &= a(d - s_p) + s_{p+1} - s_p - e \end{aligned}$$

for all p.

For any neighborhood of zero U' in $\hat{R}$ there exists a neighborhood of zero W' in R such that $aW' + W' \subseteq U'$, and such a number r that $d - s_p \in W'$ and $s_{p+1} - s_p \in W'$ for all $p \geq r$. Then $ad = a(d - s_p) + s_{p+1} - s_p - e \in a \cdot W' + W' - e \subseteq -e + U'$. From the arbitrariness of U' it follows that $ad = -e$. Hence $a(-d) = e$.

The invertibility of the element a on the left is proved analogously.

Theorem 3. *If the completion $\hat{R}$ of a topological ring R contains an identity and every element of R is invertible in R, then $[aV]_R$ is a neighborhood of zero in R for every $a \in R$ and every neighborhood of zero V in R.*

Proof. Indeed, $[V]_{\hat{R}}$ is a neighborhood of zero in $\hat{R}$. Since a is invertible in $\hat{R}$, $a[V]_{\hat{R}}$ is a neighborhood of zero in $\hat{R}$. Then $[aV]_{\hat{R}} = a[V]_{\hat{R}}$ will also be a neighborhood of zero in $\hat{R}$. Thus, $[aV]_R = R \cap [aV]_{\hat{R}}$ will be a neighborhood of zero in R.

Corollary. *If a topological ring R has a topologically nilpotent neighborhood of zero V and contains no closed one-sided ideals, then R has a countable base of neighborhoods of zero.*

Indeed, let $\{V_\alpha\}$ be some base of neighborhoods of zero in R. Without loss of generality, we may assume that all V_α are closed sets. Since for any neighborhood of zero V_α there exists such an n that $[a^{nV}]_R \subseteq [V^{n+1}]_R \subseteq [V_\alpha]_R = V_\alpha$ for $a \in V$, the collection $\{[a^{nV}]_R\}$ is a base of neighborhoods of zero in R.

Theorem 4. *If a locally bicompact ring R with identity e contains no closed one-sided ideals, then R is a field.*

Proof. If C is the component of zero in R, then C is a closed ideal, and hence either $C = 0$, or $C = R$.

Since the left annihilator of any element $a \in R$ is a closed left ideal and $e \cdot a \neq 0$, R contains no nonzero zero divisors.

If $C = R$, then, by (4) (Theorem 2), R is a finite-dimensional algebra over the field of real numbers. By Frobenius' theorem, R is either the field of real numbers, or the field of complex numbers, or the field of quaternions.

If, however, $C = 0$, then, by (4) (Lemma 4), R has a base of neighborhoods of zero that are subrings.

Let V be a bicompact neighborhood of zero, not containing e and being a ring. Since V is a ring without zero divisors, by (5) (Theorem 19), V is either radical in the sense of the Jacobson radical, or completely primary. If V were completely primary, then V would have an identity $f \neq e$. Then $(e - f)f = 0$. But this is impossible, since R contains no zero divisors. Thus, V is a radical ring in the sense of the Jacobson radical. From the fact that V is totally disconnected it follows that V is a topologically nilpotent ring. Hence V will be a topologically nilpotent set in R. By Theorem 2, every element of R is invertible in the completion $\hat{R}$. But since a locally bicompact ring is complete, $R = \hat{R}$, and therefore R is a field.

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Note: Figure translations are in progress. See original paper for figures.