A CONSTRUCTIVE CHARACTERIZATION OF ONE CLASS OF NONPERIODIC FUNCTIONS

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Abstract

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MATHEMATICS

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A CONSTRUCTIVE CHARACTERIZATION OF ONE CLASS OF NONPERIODIC FUNCTIONS

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Let $f(x) \in L_2[-1,1]$; $E_n^{(2)}[f]$ is the best approximation to $f(x)$ on $[-1,1]$ by algebraic polynomials of degree not exceeding n in the metric L_2.

Denote

$$f_h(x) = \frac{1}{\pi}\int_0^\pi f(x\cos h + \sqrt{1-x^2}\sin h\cos\theta)\,d\theta.$$

The article considers the question of the structural properties of the class of functions satisfying the condition

$$E_n^{(2)}[f] \le M/n^{s+\gamma}, \qquad n > s, \qquad n' = 1, 2, \dots, \qquad 0 < \gamma < 1,$$

where M is a constant independent of n.

Theorem 1. In order that the inequality

$$E_n^{(2)}[f] \le M/n^{s+\gamma}$$

hold, it is necessary and sufficient that

$$\left\{\int_{-1}^1 \left[\frac{d^s f_h(x)}{dx^s} - \frac{d^s f(x)}{dx^s}\right]^2 (1-x^2)^s\,dx\right\}^{1/2} \le ch^\gamma,$$

where c is a constant independent of $h > 0,\ n > s,\ 0 < \gamma < 1$.

It should be noted that the question of approximation of functions in the metric L_p by algebraic polynomials with a certain fractional-rational weight was considered by M. K. Potapov and G. K. Lebedev ([4]).

In proving the lemmas and theorems, the article uses the classical method of 2^n-summation of S. N. Bernstein ([1,2]).

Lemma 1. For the Legendre polynomial $P_k(\cos h)$, for any h and k the inequality

$$1 - P_k(\cos h) \le k^2h^2/2 \tag{1}$$

holds, and respectively

$$1 - P_k(\cos h) \ge 4k^2h^2/3\pi^3, \qquad 0 \le kh \le \pi. \tag{2}$$

Proof. We use the expression for the Legendre polynomial ([3])

$$P_k(\cos h) = 2\frac{(2k-1)!!}{(2k)!!}\cos kh + 2\frac{(2k-3)!!}{(2k-2)!!}\frac{1}{2}\cos(k-2)h + \cdots$$

$$\cdots + 2\frac{(2k-2m-1)!!}{(2k-2m)!!}\frac{(2m-1)!!}{(2m)!!}\cos(k-2m)h + \cdots$$

Since $1 = P_k(1)$, it follows that

$$1 - P_k(\cos h) = 4\frac{(2k-1)!!}{(2k)!!}\sin^2 k\frac{h}{2} + 4\frac{(2k-3)!!}{(2k-2)!!}\frac{1}{2}\sin^2(k-2)\frac{h}{2} + \cdots$$

$$\cdots + 4\frac{(2k-2m-1)!!}{(2k-2m)!!}\frac{(2m-1)!!}{(2m)!!}\sin^2(k-2m)\frac{h}{2} + \cdots \tag{3}$$

Hence, from the equality $1 = P_k(1)$, taking into account the inequality $\sin^2 x \le x^2$, we obtain (1).

On the basis of the inequality

$$\left[\frac{(2m)!!}{(2m-1)!!}\right]^2\frac{1}{2m+1} < \frac{\pi}{2} < \left[\frac{(2m)!!}{(2m-1)!!}\right]^2\frac{1}{2m} \tag{4}$$

we have

$$\frac{(2k-2m-1)!!}{(2k-2m)!!}\frac{(2m-1)!!}{(2m)!!} > \frac{2}{\pi}\frac{1}{\sqrt{(2k-2m+1)(2m+1)}} > \frac{2}{\pi(k+1)}. \tag{5}$$

Since $0 \le kh \le \pi$, it follows from (3), (5), and the inequality $\sin x \ge 2x/\pi$, where $0 \le x \le \pi/2$, that

$$1 - P_k(\cos h) \ge \frac{8h^2}{\pi^3(k+1)}\left[k^2 + (k-2)^2 + \cdots + (k-2m)^2 + \cdots\right] \ge \frac{4k^2h^2}{3\pi^3}. \tag{6}$$

Let $f(\cos\beta)\sqrt{\sin\beta}$ be a square-summable function on $[0,\pi]$. Denote

$$b_k = \frac{2k+1}{2} \int_0^\pi f(\cos\beta) P_k(\cos\beta) \sin\beta \, d\beta, \qquad a_k^2 = \frac{2b_k^2}{2k+1}. \tag{7}$$

Under these assumptions, the following assertion is valid.

Lemma 2. In order that the inequality

$$\left(\sum_{k=n}^{\infty} a_k^2 \right)^{1/2} \leq M/n^{s+\gamma} \tag{8}$$

hold, it is necessary and sufficient that

$$\left\{ \int_0^\pi \left[\frac{d^s}{d\cos\beta^s} \frac{1}{\pi} \int_0^\pi f(\cos R)\, d\theta - \frac{d^s}{d\cos\beta^s} f(\cos\beta) \right]^2 \sin^{2s+1}\beta \, d\beta \right\}^{1/2} \leq ch^\gamma, \tag{9}$$

where $\cos R = \cos\beta \cos h + \sin\beta \sin h \cos\theta, \quad n > s, \quad 0 < \gamma < 1.$

Proof. Suppose that (9) holds. From the relation

$$f(\cos\beta) \sim \sum_{k=0}^{\infty} b_{kP} k(\cos\beta),$$

applying the addition theorem for Legendre polynomials

$$P_k(\cos R) = P_k(\cos\beta) P_k(\cos h) + 2 \sum_{m=1}^{k} \frac{(k-m)!}{(k+m)!} P_k^m(\cos\beta) P_k^m(\cos h) \cos m\theta,$$

we obtain

$$\left[\frac{d^s}{d\cos\beta^s} \frac{1}{\pi} \int_0^\pi f(\cos R)\, d\theta - \frac{d^s}{d\cos\beta^s} f(\cos\beta) \right] \sin^s\beta \sim$$

$$\sim \sum_{k=s}^{\infty} [P_k(\cos h) - 1] P_k^s(\cos\beta) b_k.$$

By Parseval’ s equality, taking into account (7) and the fact that

$$\int_0^\pi \{P_k^s(\cos\beta)\}^2 \sin\beta \, d\beta = \frac{2}{2k+1} \frac{(k+s)!}{(k-s)!},$$

we shall have

$$\int_0^{\pi} \left[\frac{d^s}{d\cos\beta^s} \frac{1}{\pi} \int_0^{\pi} f(\cos R)\, d\theta - \frac{d^s}{d\cos\beta^s} f(\cos\beta) \right]^2 \sin^{2s+1}\beta\, d\beta = \sum_{k=s}^{\infty} [P_k(\cos h) - 1]^2 \frac{(k+s)!}{(k-s)!}\, a_k^2. \tag{10}$$

For any $n > s$, from condition (9) we have

$$\frac{(n+s)!}{(n-s)!} \sum_{k=n}^{2n-1} [P_k(\cos h) - 1]^2 a_k^2 \le \sum_{k=s}^{\infty} [P_k(\cos h) - 1]^2 \frac{(k+s)!}{(k-s)!}\, a_k^2 \le c^2 h^{2\gamma}.$$

Putting $h = 1/n$ and taking into account here that $1 \le kh < 2$, on the basis of inequality (2) we obtain

$$\frac{(n+s)!}{(n-s)!} \sum_{k=n}^{2n-1} a_k^2 \le \frac{(n+s)!}{(n-s)!} \sum_{k=n}^{2n-1} k^4 h^4 a_k^2 \le \frac{c_1}{n^{2\gamma}}, \qquad c_1 = \frac{9}{16}\pi^6 c^2. \tag{11}$$

Since $n > s$, for $i = 1, 2, \dots, s-1$ we have $n - i > \frac{s-i}{s} n$. Hence, also from the inequality $(n+1)(n+2)\cdots(n+s) > n^s$, it follows that

$$\frac{(n-s)!}{(n+s)!} \le \frac{s^s}{s!} \frac{1}{n^{2s}}.$$

The last inequality and (11) give

$$\sum_{k=n}^{2n-1} a_k^2 \le \frac{c_2}{n^{2s+2\gamma}}, \qquad c_2 = c_1 \frac{s^s}{s!}. \tag{12}$$

Therefore

$$\sum_{k=n}^{\infty} a_k^2 = \sum_{j=0}^{\infty} \sum_{k=2^j n}^{2^{j+1}n-1} a_k^2 \le \frac{c_2}{n^{2s+2\gamma}} \sum_{j=0}^{\infty} 2^{-2j(s+\gamma)} < \infty.$$

Taking into account the convergence of the series on the right, we obtain (8).

Let us show that (9) follows from (8). The inequality holds

$$\sum_{k=s}^{\infty} [P_k(\cos h) - 1]^2 \frac{(k+s)!}{(k-s)!} a_k^2 \le 2^s \sum_{k=s}^{\infty} [P_k(\cos h) - 1]^2 k^{2s} a_k^2.$$

Represent the following sum by two terms:

$$\sum_{k=s}^{\infty}[P_k(\cos h)-1]^2k^{2s}a_k^2=\sum_{k=s}^{n-1}+\sum_{k=n}^{\infty},\qquad n=\left[\frac{1}{h}\right].$$

Put

$$\gamma_k=\sum_{m=k}^{\infty}m^{2s}a_m^2\le M_2/k^{2\gamma}. \tag{13}$$

Inequality (13) follows from (12) and is equivalent to (8). Taking inequalities (1) and (13) into account, we shall have

$$\sum_{k=s}^{n-1}[P_k(\cos h)-1]^2k^{2s}a_k^2\le\sum_{k=1}^{n}k^4h^4k^{2s}a_k^2=h^4\sum_{k=1}^{n}k^4(\gamma_k-\gamma_{k-1})\le$$

$$\le h^4\{\gamma_1+(2^4-1^4)\gamma_2+(3^4-2^4)\gamma_2+\cdots+[n^4-(n-1)^4]\gamma_n\}\le$$

$$\le 6h^4\sum_{k=1}^{n}k^3\gamma_k\le 6M_2h^4\sum_{k=1}^{n}k^{3-2\gamma}\le M_3h^{2\gamma}. \tag{14}$$

Since $|P_k(\cos h)|\leqslant 1$, we have

$$\sum_{k=n}^{\infty}[P_k(\cos h)-1]^2k^{2s}a_k^2\leqslant 4\sum_{k=n}^{\infty}k^{2s}a_k^2\leqslant 4M_2h^{2\gamma}. \tag{15}$$

Combining the estimates (14) and (15), we obtain (9).
Theorem 1 follows from Lemma 2, if we put $x=\cos\beta$ and take into account that

$$\left\{\int_{-1}^{1}\left[f(x)-\sum_{k=0}^{n-1}b_kP_k(x)\right]^2dx\right\}^{1/2}=\left(\sum_{k=n}^{\infty}a_k^2\right)^{1/2}=E_n^{(2)}[f].$$

Next we shall prove a theorem on the absolute convergence of the series in Legendre polynomials, which is an analogue of S. N. Bernstein' s theorem on the absolute convergence of the trigonometric Fourier series (1).

Theorem 2. If the function $f(x)$ satisfies the condition $|f_h(x)-f(x)|\leqslant ch^{\gamma}$, $-1\leqslant x\leqslant 1$, $h>0$, $\gamma>1/2$, then the series $\sum_{k=2}^{\infty}|a_k|$ converges.

Proof. For $s=0$, from (10) we obtain

$$\int_0^\pi \left[\frac{1}{\pi}\int_0^\pi f(\cos R)\,d\theta - f(\cos\beta)\right]^2 \sin\beta\,d\beta = \sum_{k=0}^{\infty}[P_k(\cos h) - 1]^2 a_k^2.$$

Putting $x = \cos\beta$, from the condition of Theorem 2 we shall have

$$\sum_{k=0}^{\infty}[P_k(\cos h) - 1]^2 a_k^2 \leqslant 2c^2 h^{2\gamma}.$$

Choose an arbitrary natural number N and put $h = 1/N$. Then

$$\sum_{k>N/2}^{N}[P_k(\cos h) - 1]^2 a_k^2 \leqslant 2c^2 N^{-2\gamma}.$$

Hence, and from inequality (2),

$$\sum_{k>N/2}^{N} a_k^2 \leqslant 2c_1 N^{-2\gamma}.$$

In particular, if we put $N = 2^\nu$, $\nu = 1, 2, ...$, then

$$\sum_{k=2^{\nu-1}+1}^{2^\nu} a_k^2 \leqslant 2c_1 2^{-2\gamma\nu}.$$

Hence, and from Bunyakovsky' s inequality,

$$\sum_{k=2^{\nu-1}+1}^{2^\nu} |a_k| \leqslant \left(\sum_{k=2^{\nu-1}+1}^{2^\nu} a_k^2\right)^{1/2} \left(\sum_{k=2^{\nu-1}+1}^{2^\nu} 1\right)^{1/2} \leqslant \sqrt{c_1}\, 2^{\nu(1/2-\gamma)}.$$

Consequently,

$$\sum_{k=2}^{\infty} |a_k| = \sum_{\nu=1}^{\infty} \sum_{k=2^{\nu-1}+1}^{2^\nu} |a_k| \leqslant \sqrt{c_1} \sum_{\nu=1}^{\infty} 2^{(1/2-\gamma)\nu} < \infty.$$

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Note: Figure translations are in progress. See original paper for figures.