ON THE THEORY OF CANONICAL SYSTEMS OF DIFFERENTIAL EQUATIONS OF POSITIVE TYPE WITH PERIODIC COEFFICIENTS

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Abstract

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ON THE THEORY OF CANONICAL SYSTEMS OF DIFFERENTIAL EQUATIONS OF POSITIVE TYPE WITH PERIODIC COEFFICIENTS

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In this note we use the notation and results of the works ($^{1-3}$). All terminology has been borrowed by us from (^{3}).

We consider a real linear differential system with periodic coefficients of the form

$$dx/dt = JH(t)x \qquad (H(t+T) = H(t)), \tag{1}$$

where x is an n-dimensional ($n = 2m$) vector function; $H(t)$ is a symmetric matrix function with elements from $L_1(0,T)$; $J = \operatorname{diag}\{J_1, \dots, J_1\}$, where

$$J_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Let $X(t)$ be the matriciant of equation (1), i.e., an absolutely continuous matrix function satisfying equation (1) and the initial condition $X(0) = I$. We note that this matrix is symplectic for all $t \in [0,T]$. It can be represented in the form

$$X(t) = e^{JQ(t)} \qquad (Q = Q^*).$$

From the absolute continuity of $X(t)$ follows the absolute continuity of $Q(t)$.

Equation (1) is called an **equation of positive type** if $H(t) \geqslant 0$ $(0 \leqslant t \leqslant T)$ and

$$\int_0^T H(t)\,dt > 0.$$

Following R. Bott (^{5}), we shall call the matriciant of an equation of positive type a **positively directed symplectic curve**.

Main theorem. *Every stable equation* of the form (1) *of positive type can be continuously deformed into a split system of equations of the form*

$$dx_k/dt = J_1 H_k(t) x_k(t) \qquad (k = 1, 2, \dots, m) \tag{2}$$

$$(x_1, \dots, x_m \text{ are 2-dimensional vectors})$$

with preservation (in the process of deformation) of stability and positivity.

The proof of this theorem is based on the representation of the Hamiltonian H of equation (1) through the eigenvalues and eigenvectors of the matrix $-iJQ(t)$.

If, for some $t = t_0$, the matrix $-iJQ(t)$ has a simple spectrum, then we shall call t_0 a **simple point**, and otherwise a **multiple point**.

In a neighborhood of a simple point the matrix $-iJQ$ has p pairs $(\omega, -\omega + 2k\pi)$ of real eigenvalues, q pairs $(\omega, -\omega = \bar{\omega})$ of numbers of imaginary eigenvalues and r quadruples $(\omega, \bar{\omega}, -\bar{\omega} + 2k\pi, -\omega + 2k\pi)$ of complex eigenvalues $(2p + 2q + 4r = n)$. The corresponding n eigenvectors can be normalized and numbered so that the relation holds:

$$-iS^* J S = \hat{J}.$$

Here S is the matrix whose columns are the eigenvectors of the matrix $-iJQ(t)$, and

$$\hat{J} = \operatorname{diag}\left\{ \underbrace{\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \dots}_{p\ \text{blocks}};\ \underbrace{\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \dots}_{q\ \text{blocks}};\ \underbrace{\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}, \dots}_{r\ \text{pairs of blocks}} \right\}.$$

Theorem 1. In a neighborhood of a nonsingular point t_0, the matrix H is representable in the form*

$$H = S^* \left(\frac{d\Omega}{dt} \hat{J} + E \circ K \right) S, \tag{3}$$

where $\Omega = \operatorname{diag}\{\omega_1, \dots, \omega_n\}$ is the diagonal form of the matrix $-iJQ(t)$,

$$E = \left\| i(e^{i(\bar{\omega}_j - \omega_k)} - 1) \right\|_{k,j=1}^{n}, \qquad K = -iS^* J\, dS/dt$$

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* By a stable equation we mean an equation with a strongly stable monodromy matrix $X(T)$.

is a skew-Hermitian matrix**. Moreover,

$$e^{i\bar{\Omega}} = -Pe^{i\Omega}P, \qquad \bar{S} = SP, \qquad \bar{K} = -PKP,$$

where P is a permutation matrix:

$$P = \operatorname{diag}\left\{ \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \dots;}_{p \text{ blocks}} \; I_{2q}; \underbrace{\begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \dots}_{q \text{ blocks}} \right\}$$

(I_s is the identity matrix of order s).

Conversely, suppose that in a neighborhood of the point t_0 the following are given:

1°. A collection of functions $\omega = \omega_k(t)$ $(k = 1, 2, \dots, n)$, pairwise distinct for every t, and such that the matrix $\Omega = \operatorname{diag}\{\omega_1, \dots, \omega_n\}$ has the property

$$e^{i\bar{\Omega}} = -Pe^{i\Omega}P.$$

2°. An arbitrary skew-Hermitian matrix $K(t)$, for which

$$\bar{K} = -PKP.$$

3°. A constant matrix S_0, for which

$$-iS_0^* J S_0 = \hat{J}$$

and

$$\bar{S}_0 = S_0 P.$$

Let $S(t)$ be the solution of the equation

$$dS/dt = S\hat{J}K,$$

satisfying the initial condition

$$S(t_0) = S_0.$$

If the Hamiltonian H of equation (1) has, in a neighborhood of the point t_0, the form (3), then for the matrizant $X(t)$ in the same neighborhood the matrix $e^{i\Omega}$ is a diagonal form, and the matrix S is the matrix of eigenvectors.

The proof is carried out by direct calculation of the matrix

$$H = -J\frac{dX}{dt}X^{-1}$$

using the formula of Yu. L. Daletskii–S. G. Krein (4) for finding

$$\frac{dX}{dt} = \frac{d}{dt} e^{JQ(t)}.$$

From this, as a consequence, we immediately obtain a spectral characterization of a positively directed symplectic curve.

In order that a symplectic curve $X(t) = e^{JQ(t)}$ with pairwise distinct eigenvalues be positively directed, it is necessary and sufficient that the matrix

$$\mathfrak{H} = \frac{d\Omega}{dt} \hat{J} + E \circ K$$

be ≥ 0 for all t, and on some set of positive measure $\mathfrak{H} > 0$.

This criterion and representation (3) make it possible to construct a deformation of system (1) into the split system (2).

Without loss of generality one may assume that on the interval $[0, T]$ there is only a finite set of multiple points

$$t_1 < t_2 < \cdots < t_N.$$

The points t_k $(k = 1, 2, \dots, N)$ divide $[0, T]$ into a number of subintervals without multiple points.

* Here $E \circ K$ is the Hadamard (elementwise) product of E and K,

** The last relation is equivalent to the relation $dS/dt = S\hat{J}K$.

For convenience of notation, we shall assume that on some fixed interval the eigenvalues ω_k $(k = 1, 2, \dots, n)$ are numbered as follows: first the p real pairs are numbered, then the q purely imaginary pairs, and then the r complex quadruples. Write H in the form (3) and set

$$R = \operatorname{diag}\{\underbrace{1, \dots, 1}_{2p} \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \dots, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}}_{q+2r \text{ blocks}},$$

$$K_1 = K \circ R, \qquad K_2 = K - K_1 \qquad (K_1^* = -K_1,\ K_2^* = -K_2).$$

Rewrite the matrix H in the form $H = S^*(\mathfrak{H}_1 + E \circ K_2)S$, where

$$\mathfrak{H}_1 = \frac{d\Omega}{dt} \hat{J} + E \circ K_1 \geq 0.$$

The deformation of system (1) into (2) on each selected interval is carried out in two steps.

I. We deform the eigenvectors of the matrix $-iJQ$ (without touching its eigenvalues) so that $\mathfrak{H}$ is simplified. Namely, put $K_\alpha = K_1 + (1-\alpha)K_2$ $(0 \le \alpha \le 1)$. Then $\mathfrak{H}_\alpha = \mathfrak{H}_1 + (1-\alpha)E \circ K_2 = (1-\alpha)\mathfrak{H} + \alpha\mathfrak{H}_1 \ge 0$. For $\alpha = 1$ we arrive at the block-diagonal matrix* $\mathfrak{H}_1$: to a real eigenvalue ω there corresponds the diagonal element ω', and to a complex-conjugate pair $(\omega, \bar{\omega})$ there corresponds a diagonal block of the form

$$\begin{pmatrix} i(e^{2\operatorname{Im}\omega}-1)k_+ & \omega' \\ \bar{\omega}' & i(e^{-2\operatorname{Im}\omega}-1)k_- \end{pmatrix}.$$

II. We deform the eigenvalues of the matrix $-iJQ$ so that the complex eigenvalues $\pm\omega$, $\pm\bar{\omega}$ pass onto the real axis. At the same time we deform the eigenvectors so that $\mathfrak{H}$ remains nonnegative in the process of deformation. Namely, if ω_0 leaves the real axis at $t = t_k$ and returns to it at $t = t_l$, put

$$\omega_0(t;\beta) = (1-\beta)\omega_0(t) + \beta\omega_0(t_l), \qquad k_\pm(t;\beta) = \frac{(1-\beta)(e^{\pm 2\operatorname{Im}\omega_0}-1)}{e^{\pm 2\operatorname{Im}(1-\beta)\omega_0}-1}\,k_\pm$$

$$(0 \le \beta \le 1;\ t_k \le t \le t_l).$$

For $\beta = 1$ system (1) on the indicated interval passes into (2). For $t > t_l$ the real increasing differentiable function $\omega_0(t)$ may be deformed arbitrarily, preserving reality, increase, and differentiability, taking care only that for all $\beta \in [0,1]$ continuity be satisfied at the junction: $\omega_0(t_l - 0) = \omega_0(t_l + 0)$. Repeating this process, we arrive at (2). A detailed study of the features arising at multiple points shows that in the process of deformation $H(t)$ remains summable.

Corollary. *Two positively directed symplectic curves from one stability region can be deformed one into the other with preservation of positive directedness, without leaving the indicated stability region.*

Consider the spectrum of the monodromy matrix $X(T)$. Moving along the upper semicircle of the unit circle from 1 to -1, we shall write 0 when we encounter a multiplier of the first kind, and 1 when we encounter a multiplier of the second kind (each of the numbers $0, 1$ is written as many times as is the multiplicity of the corresponding multiplier). As a result we obtain an n-digit binary number –the **multiplier type** of the corresponding system (1).

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* It does not follow from this that system (1) has already been reduced to the form (2). It can be shown that the matrix S in this case has not yet split into second-order blocks.

Theorem 2. The index of the stability domain containing system (1) is equal to the number of pairs of characteristic numbers of the antiperiodic boundary-value problem

$$dx/dt = \lambda JHx, \qquad x(T) = -x(0), \tag{4}$$

lying between $\lambda = 0$ and $\lambda = 1$, minus the number of ones in the binary notation of the multiplier type of this equation.

For $n = 2$, the assertion of the theorem is easily obtained from the considerations of (1). In the case of arbitrary n, it remains to note that under the deformation described above the multiplier type, the index, and the number of characteristic numbers of the boundary-value problem (4) between $\lambda = 0$ and $\lambda = 1$ do not change.

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Note: Figure translations are in progress. See original paper for figures.