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MATHEMATICS

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Abstract

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MATHEMATICS

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INVARIANT SUBSPACES OF THE SHIFT OPERATOR AND WEAKLY INVERTIBLE ELEMENTS IN L^∞

(Presented by Academician V. I. Smirnov on 9 VII 1965)

1°. Let L^p be the space of all measurable complex functions defined and summable to the power p , $1 \leq p < \infty$, on the interval $[0, 2\pi]$, equipped with the usual norm; L^∞ is the space of all essentially bounded functions with the ess sup-norm. We say "functions," instead of "classes of functions differing on a set of Lebesgue measure zero," wherever this cannot cause ambiguity. Let C be the set of functions v continuous on $[0, 2\pi]$ and satisfying the condition $v(0) = v(2\pi)$; let H^∞ be the subspace of L^∞ consisting of functions v of the form $v(\theta) = h_v(\theta)$, $\theta \in [0, 2\pi]$, where h_v are the boundary values of a complex function analytic and bounded in the disk $\{z : |z| < 1\}$.

The space L^∞ may be regarded as a space of operators acting in L^2 . Namely, an element v , $v \in L^\infty$, generates the operator $f \rightarrow vf$, $f \in L^2$. By the (s) -topology on L^∞ we shall henceforth mean the strong operator topology of the space L^∞ . We shall say that functions v_n , $v_n \in L^\infty$, $n = 1, 2, \dots$, $(*)$ -converge to a function v , $v \in L^\infty$, if $\|v_n\|_\infty \leq K$, $n = 1, 2, \dots$, for some K , and

$$\lim_n v_n(\theta) = v(\theta)$$

for almost all θ , $\theta \in [0, 2\pi]$. A subspace M , $M \subset L^\infty$, is called $(*)$ -closed if from $v_n \in M$, $v \in L^\infty$, and $(*)$ - $\lim_n v_n = v$, it follows that $v \in M$.

Define the shift operator S in L^∞ by the formula

$$(Sv)(\theta) = e^{i\theta}v(\theta), \quad \theta \in [0, 2\pi], \quad v \in L^\infty.$$

In this note we study invariant subspaces M of the operator S (i.e. such that $SM \subset M$), closed in the following two senses: a) in the sense of the (s) -topology, b) in the sense of $(*)$ -convergence, as well as questions of weighted approximation with respect to both types of convergence. The second type of convergence is of interest, in particular, because it guarantees strong convergence of the operators $v_n(U)$ to $v(U)$ for any unitary operator U with absolutely continuous spectrum (see, for example, ^(2, 3)).

2°. If u is a function on $[0, 2\pi]$ and X is a set of functions, then put

$$uX = \{g : g(\theta) = u(\theta)f(\theta), \theta \in [0, 2\pi], f \in X\}.$$

Theorem 1. Let M be a $(*)$ - or (s) -closed subspace of L^∞ , invariant with respect to the operator S (i.e. such that $SM \subset M$). Then one of the following two assertions holds:

either 1) $S^{-1}M \subset M$, and then there exists a measurable (unique up to a set of Lebesgue measure zero) set E , $E \subset [0, 2\pi]$, such that $M = \chi_E L^\infty$, where χ_E is the characteristic function of E ;

or 2) $S^{-1}M \not\subset M$, and then $M = FH^\infty$, where $F \in L^\infty$, $|F| = 1$ almost everywhere on $[0, 2\pi]$, and such an F is unique up to a multiplicative constant.

Remark 1. Of course, subspaces of both types are $(*)$ -closed (and hence also (s) -closed).

Remark 2. Theorem 1 is quite analogous to the Helson-Lowdenslager theorem on invariant subspaces of the shift operator in L^2 .

3°. Let $C_A = C \cap H^\infty$ and let \mathcal{P}_A be the set of polynomials from H^∞ . (In general, by a polynomial in what follows is meant a trigonometric polynomial; we denote their totality by \mathcal{P} ; then $\mathcal{P}_A = \mathcal{P} \cap C_A$.) Suppose, further, that $v \in L^\infty$; from the theorem of A. N. Kolmogorov (see, for example, ⁽¹⁾, p. 76) it follows that the element $S^{-1}v$ is then and only then the L^2 -limit of some sequence $(p_n v)_{n=1}^\infty$, $p_n \in \mathcal{P}_A$, when $\log |v| \notin L^1$. The question arises: can this sequence be chosen to be $(*)$ -convergent? It turns out that, generally speaking, this is not so, unless some smoothness of the function v is assumed.

Let f be a measurable real-valued function on $[0, 2\pi]$. Put

$$m_f(\theta) = \lim_{\delta \rightarrow +0} \operatorname{ess\,sup}_{\theta - \delta < t < \theta + \delta} f(t), \quad \theta \in [0, 2\pi].$$

To define $m_f(0)$ and $m_f(2\pi)$ we extend f 2π -periodically to the whole real axis.

Theorem 2. Let $v \in L^\infty$. The following assertions are equivalent:

- 1) There exists a sequence $(p_n)_{n=1}^\infty$, $p_n \in \mathcal{P}_A$, $n = 1, 2, \dots$, such that $\lim_n p_n v = S^{-1}v$ in the norm of L^∞ .
- 2) There exists a sequence $(q_n)_{n=1}^\infty$, $q_n \in \mathcal{P}_A$, $n = 1, 2, \dots$, such that $(*)$ - $\lim_n q_n v = S^{-1}v$.
- 3) $m_{\log |v|} \notin L^1$.
- 4) $\log |v| \notin L^1$ and there exists an upper semicontinuous function m such that $m(\theta) \geq \log |v(\theta)|$ for almost all θ , $\theta \in [0, 2\pi]$, and

$$\int_0^{2\pi} m \, d\theta = -\infty.$$

If v is continuous, then 1)–4) are equivalent to the condition

$$5) \log |v| \notin L^1.$$

Remark. Condition 3), in particular, is satisfied if $\log |v| \notin L^1$ and $m_{\log |v|} \leq \log |v| + K$, where K is some constant (see Lemma 2 below). The latter inequality may be interpreted as the absence of “too large jumps” of the function $\log |v|$.

In the proof of Theorem 2 the following proposition is used.

Lemma 1. Let $v \in L^\infty$ and $f \in L^\infty$; put $\|f\|_v = \|fv\|_\infty$. Define the functional φ on H^∞ by the formula $\varphi(u) = h_u(0)$, where $u \in H^\infty$ and h_u is the bounded analytic function in the disk $\{z : |z| < 1\}$ whose boundary values coincide with u . Then: 1) the functional φ is discontinuous in the norm $\|\cdot\|_v$ on the set C_A if and only if $m_{\log |v|} \notin L^1$; 2) the functional φ is discontinuous in the norm $\|\cdot\|_v$ on the set H^∞ if and only if $\log |v| \notin L^1$.

4°. Following G. Shapiro (⁴), we shall call an element v , $v \in L^\infty$, weakly $(*)$ -invertible if there exist p_n , $p_n \in \mathcal{P}_A$, $n = 1, 2, \dots$, such that

$$(*)\text{-}\lim_n p_n v = 1$$

(here only polynomials p from \mathcal{P}_A are considered, since for them $pv = p(S)v$, where $p(S)$ is understood as an operator polynomial). To describe all weakly $(*)$ -invertible elements of L^∞ , we first formulate the following lemma.

Lemma 2. Let f be a measurable real-valued function on $[0, 2\pi]$ and $f(\theta) \leq K$ for almost all θ , $\theta \in [0, 2\pi]$, and for some K .

The following conditions are equivalent:

- 1) There exist a constant K_1 and an upper semicontinuous function m such that

$$m(\theta) \leq f(\theta) \leq m(\theta) + K_1$$

almost everywhere.

- 2) there exists a constant K_1 such that

$$m_f(\theta) \leq f(\theta) + K_1 \quad \text{almost everywhere.}$$

- 3) There exist a constant K_1 and a sequence of real polynomials $(p_n)_{n=2}^\infty$, $p_n \in \mathcal{P}$, $n = 1, 2, \dots$, such that: $\lim_n p_n(\theta) = f(\theta)$ for almost all θ , $\theta \in [0, 2\pi]$;

$f(\theta) \leq p_n(\theta) + K_1$ for almost all θ , $\theta \in [0, 2\pi]$, and for all n , $n = 1, 2, \dots$, and $p_n(\theta) \leq 2K$, $\theta \in [0, 2\pi]$.

Theorem 3. Let $v \in L^\infty$. The function v is weakly $(*)$ -invertible if and only if either 1) $\log|v| \in L^1$, v is an outer function from H^∞ (see, for example, ⁽¹⁾, p. 92) and for the function $\log|v|$ one of the conditions of Lemma 2 is fulfilled; or 2) $\log|v| \notin L^1$, for the function $\log|v|$ one of the conditions of Lemma 2 is fulfilled and $v(\theta) \neq 0$ for almost all θ , $\theta \in [0, 2\pi]$.

5°. Let $v \in L^\infty$. A natural continuation of Theorems 2 and 3 is the description of those functions g , $g \in L^\infty$, which can be $(*)$ -approximated by some sequence $(p_n v)_1^\infty$, $p_n \in \mathcal{P}_A$, $n = 1, 2, \dots$, i.e. the description of the set $M_v = \{g : g \in L^\infty \text{ and there exist } p_n, n = 1, 2, \dots, p_n \in \mathcal{P}_A, \text{ such that } (*)\text{-}\lim_n p_n v = g\}$. Denote by D the class of functions introduced by V. I. Smirnov: this is the class of measurable functions v on $[0, 2\pi]$ which are boundary values of functions h_v of the form $h_v = G \cdot F$, where G is an outer analytic function (not necessarily from H^1) and F is an inner function from H^∞ (see ⁽¹⁾).

Theorem 4. Let $v \in L^\infty$.

1) If $m_{\log|v|} \in L^1$, then

$$M_v = \{hv : h \in D \text{ and } hm_{|v|} \in L^\infty\}.$$

2) If $m_{\log|v|} \notin L^1$, then

$$M_v = \{hv : h \text{ is a measurable function on } [0, 2\pi] \text{ and } hm_{|v|} \in L^\infty\}.$$

Remark. The class D of V. I. Smirnov arises naturally in questions of approximation by analytic functions ⁽¹⁾, and Theorem 4 adjoins the investigations of G. Ts. Tumarkin on the convergence of sequences of boundary values of analytic functions ⁽⁵⁾.

Corollary 1. Let $v \in L^\infty$ and $m_{\log|v|} \in L^1$. Then $M_v = vH^\infty$ if and only if $1/m_{|v|} \in L^\infty$ (i.e. when the norms $\|\cdot\|_\infty$ and $\|\cdot\|_v$ are equivalent on \mathcal{P}_A).

Corollary 2. Let $v \in L^\infty$ and $\log|v| \in L^1$ (therefore, $v = v_0 v_1$, where $v_0 \in L^\infty$, $|v_0| = 1$ almost everywhere, and v_1 is an outer function from H^∞ ⁽¹⁾). Then $M_v = v_0 H^\infty$ if and only if $m_{|v|}/|v| \in L^\infty$, i.e. one of the conditions of Lemma 2 is fulfilled for the function $\log|v|$.

Corollary 3. Let $v \in L^\infty$ and $m_{\log|v|} \notin L^1$. Suppose further that \bar{v} is a function, a certain representative of the element v , and $E = \{\theta : \theta \in [0, 2\pi], \bar{v}(\theta) \neq 0\}$.

1) In order that $M_v = vL^\infty$, it is necessary and sufficient that the condition $m_{|v|}(\theta) \geq \delta > 0$ hold on the set E for some fixed δ .

2) In order that $M_v = \chi_E L^\infty$, it is necessary and sufficient that the condition $m_{|v|}/|v| \in L_E^\infty$ hold, where $(m_{|v|}/|v|)(\theta) = m_{|v|}(\theta)/|v(\theta)|$, $\theta \in E$, and L_E^∞ is the restriction of L^∞ to the set E .

6°. Consideration of S -invariant subspaces of L^∞ , closed in the norm of this space, encounters great difficulties. Some idea of them is perhaps given by the following theorem (it also supplements Lemma 1 and shows that in Theorem 4 one cannot replace $(*)$ -convergence by uniform convergence, even for the simplest functions $v, v \in L^\infty$).

Theorem 5. If E is a measurable set, $E \subset [0, 2\pi]$, of positive Lebesgue measure, then $\overline{\chi_E H^\infty} \neq \chi_E L^\infty$, where the bar denotes closure in the norm of L^∞ . (Let us note, however, that if the measure of the complement $CE = [0, 2\pi] \setminus E$ is also positive, then it follows from Lemma 1 that all functions $\chi_E f$ belong to $\overline{\chi_E H^\infty}$, where $f \in C$.)

Proof. To verify the theorem, it is enough to observe that there exists a multiplicative functional on the Banach algebra $\overline{\chi_E H^\infty}$ which cannot be extended to a multiplicative functional in $\chi_E L^\infty = L_E^\infty$. Let $\theta_0, \theta_0 \in E$, be a point of density of the set E , and let $z_k = r_k e^{i\theta_0}$, $0 < r_k < 1$, $k = 1, 2, \dots$, be a sequence of points of the disk $\{z : |z| < 1\}$ tending to $e^{i\theta_0}$, and such that

$$\sum_{k=1}^{\infty} (1 - r_k) < \infty.$$

Let

$$\Lambda_{\theta_0} = \{f : f \in H^\infty \text{ and there exists } \lim_k h_f(z_k) = \varphi(f)\}.$$

Recall that h_f is the bounded analytic function in the unit disk whose boundary values are the function f . It is clear that the functional φ is continuous on the set Λ_{θ_0} in the norm $\|\cdot\|_{\chi_E}$. Therefore there exists a multiplicative functional Φ in $\overline{\chi_E H^\infty}$ such that $\Phi \neq 0$ and $\Phi(\chi_E f) = 0$, if $\varphi(f) = 0$. If B is the Blaschke product (see (1)) with zeros z_k , $k = 1, 2, \dots$, then $\Phi(\chi_E B) = 0$, and the element $\chi_E B$ is invertible in L_E^∞ . The theorem is proved.

Remark. It is easy to verify that Theorems 1-4 remain valid if everywhere one replaces $(*)$ -convergence by (s) -convergence or by convergence in the weak topology of the space L^∞ induced by the duality between L^1 and L^∞ .

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