ON SPATIAL GROUPS OF SIMILARITY SYMMETRY

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ON SPATIAL GROUPS OF SIMILARITY SYMMETRY

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1. In 1960 A. V. Shubnikov introduced into the Soviet literature the concept of groups of similarity symmetry (^{1}). A refinement of this concept, a more complete study of two-dimensional groups of similarity symmetry, and the transfer to them of the ideas of antisymmetry (^{2}) and of some of its generalizations (^{3}) were carried out in (^{4}).

A similarity transformation P with similarity coefficient k in an n-dimensional Euclidean space is defined by the property $A'B' = k \cdot AB$, where $A' = P(A)$, $B' = P(B)$, and the points A and B are arbitrary. If $k \neq 1$, then P has a unique fixed point O and is the product of a homothety K with center O and coefficient k^* by a motion s (of the first or second kind) preserving the fixed point O; moreover, K and s commute. These facts were proved in (^{5}) for $n \leq 3$ and are proved analogously for any n.

The point O, fixed under a given similarity transformation P with $k \neq 1$, will be called its singular point (^{4}). To it converges any sequence of points $A_1, A_2, \dots, A_n, \dots$, where, for $i = 1, 2, \dots$, either $A_{i+1} = P(A_i)$ (if $k < 1$) or $A_{i+1} = P^{-1}(A_i)$. All nonidentity transformations of the cyclic group $\{P\}$ have O as their singular point.

We shall call an n-dimensional group of similarity symmetry a group of similarity transformations in n-dimensional space having the following properties: a) it contains at least one transformation P with coefficient $k \neq 1$; b) at least one point of the space is isolated in the infinite class of its images under all transformations of the given group. The singular point of the transformation P will be called a singular point of the group of similarity symmetry.

The uniqueness of a singular point for any group of similarity symmetry is not included in the definition. It follows from the theorem expressing the basic property of a group of similarity symmetry.

Theorem. *Every transformation of an n-dimensional group of similarity symmetry leaves fixed any of its singular points.*

This theorem was proved in (4) for $n = 2$. There it was also shown that the derivation of all two-dimensional groups of similarity symmetry (and antisymmetry) can be reduced to the study of groups of symmetry and antisymmetry of rods, better known in geometrical crystallography (6–8).

The method of proof given for $n = 2$ (4) of the basic property of a group of similarity symmetry does not carry over directly to the case $n \geq 3$. In the present work this theorem is proved completely for the three-dimensional case. The method of proof proposed here already carries over completely to any n-dimensional case.

2. Let us set out some preliminary considerations for $n = 3$. As indicated in item 1, if P is any similarity transformation with coefficient $k \neq 1$, then $P = Ks$, where K is a homothety with center O and coefficient k (which we shall briefly denote by the symbol: $K \sim O, k$), and s is a motion, with $s(O) = O$. Corresponding to all types of motions s, the transformation P has four types:

1) a homothety $K \sim O, k$;
2) a screw motion (1,4) $L = Kv$, where v is a rotation through an angle φ about the axis* with directing unit vector $\mathbf{l}$ (φ is the magnitude of the oriented angle, whose orientation is coordinated with the direction of the vector $\mathbf{l}$); in brief notation: $v \sim \mathbf{l}, \varphi$ (at the same time $v \sim -\mathbf{l}, -\varphi$); $L \sim O, k, \mathbf{l}, \varphi$ ($L \sim O, k, -\mathbf{l}, -\varphi$);
3) a homothetic reflection (4) $M = Km$, where m is reflection in a plane*; $M^2 = K^2 = K_1 \sim O, k^2$;
4) a screw reflection (4) $\bar{L} = K\bar{v}$, where $\bar{v} = vm = mv$ is a rotatory reflection; $\bar{L} = Lm$, $\bar{L}^2 = L^2 = L_1 \sim O, k^2, \mathbf{l}, 2\varphi$.

For the proof of the theorem we shall need two lemmas on screw motions.

Lemma 1. *Let $L \sim O, k, \mathbf{l}, \varphi$ and $L_1 \sim O, k, \mathbf{l}_1, \varphi$; then for any point M and its $(L_1^{-1}L)$-image M_1 the formula holds*

$$MM_1 \leqslant 2 \cdot OM \cdot \widehat{(\mathbf{l}, \mathbf{l}_1)}. \tag{1}$$

Indeed, decompose each of the screw rotations into a homothety and a rotation:

$$L = Kv, \qquad L_1 = Kv_1 \qquad (K \sim O, \varphi;\ v \sim \mathbf{l}, \varphi;\ v_1 \sim \mathbf{l}_1, \varphi).$$

Then $L_1^{-1}L = v_1^{-1}v = v_2 \sim \mathbf{l}_2, \varphi_2$, with $\varphi_2 \leqslant 2\widehat{(\mathbf{l}, \mathbf{l}_1)}$ by the rule, known in the classical theory of symmetry, for multiplying rotations (2,6). By obvious geometric considerations $MM_1 \leqslant OM \cdot \varphi_2$, whence formula (1) follows.

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* $M' = K(M)$ is defined by the property: $\overrightarrow{OM'} = k \cdot \overrightarrow{OM}$.

Lemma 2. *Let* $L \sim O, k, \mathbf{l}, \varphi$ *and* $L_1 \sim O_1, k, \mathbf{l}, \varphi$*; then for any point* M *and its* (L_1^{-1})*-image* M_1 *the formula holds*

$$MM_1 \leqslant (1 + 1/k) \cdot OO_1. \tag{2}$$

In fact, let the point $M^* = L(M)$; then $M_1 = L_1^{-1}(M^*)$; further, let t be translation by the vector $\overrightarrow{O_1O}$; then $L_1^{-1} = t^{-1}L^{-1}t$; let $N^* = t(M^*)$, then

$$M^*N^* = OO_1, \tag{2_1}$$

and, finally, let $N = L^{-1}(N^*)$, then

$$MN = \frac{1}{k} \cdot M^*N^*, \tag{2_2}$$

while $M_1 = t^{-1}L^{-1}t(M^*) = t^{-1}L^{-1}(N^*) = t^{-1}(N)$, i.e. $N = t(M_1)$; hence

$$NM_1 = OO_1. \tag{2_3}$$

Formula (2) follows from the triangle inequality and formulas (2_1)–(2_3).

3. We pass to the proof of the theorem. Let a three-dimensional group of similarity symmetry be given, and let P be an arbitrary transformation of it with coefficient $k \neq 1$ (without loss of generality, we take $k < 1$), O its fixed point, and Q any other transformation from the given group. We assert that $Q(O) = O$.

We shall prove this by contradiction: the assumption that the point $O_1 = Q(O)$ does not coincide with O contradicts discreteness of the similarity symmetry group—the requirement b) in its definition.

Main case. $P = L \sim O, k, \mathbf{l}, \varphi$ (a screw motion). Let the point $O_1 = Q(O)$ be distinct from O; then $P_1 = QPQ^{-1}$ is a screw motion $L_1 \sim O_1, k, \mathbf{l}_1, \varphi$, where the vector $\mathbf{l}_1 = \pm Q(\mathbf{l})$ **.

Denote: $L_2 = LL_1L^{-1}, \ldots, L_{n+1} = LL_{nL}^{-1}, \ldots$; for $i = 2, 3, \ldots$ each $L_i \sim O_i, k, \mathbf{l}_i, \varphi$, where $O_i = L(O_{i-1})$, $\mathbf{l}_i = L(\mathbf{l}_{i-1})$. By the choice of the point O, the distance $OO_n \to 0$ as $n \to \infty$.

The sequence of vectors $\mathbf{l}_1, \mathbf{l}_2, \ldots, \mathbf{l}_n, \ldots$ is not convergent, but it is bounded (all the vectors $\mathbf{l}_i$ are unit vectors), and, by the Bolzano-Weierstrass principle, one can extract from it a convergent subsequence $\mathbf{l}_{i_1}, \mathbf{l}_{i_2}, \ldots, \mathbf{l}_{i_n}, \ldots$ (in general, not converging to the vector $\mathbf{l}$).

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[*] Passing through the point O.

[**] The minus sign is put if Q is a transformation of the 2nd kind, since in this case the orientation of the angle φ in the screw motion L_1 is not coordinated with the direction of the vector $Q(\mathbf{l})$.

We now choose an arbitrary point M. We assert that for every $\varepsilon > 0$ there exists a natural number n such that for any natural m, from the condition

$$M' = L_{i_{n+m}}^{-1} L_{i_n}(M) \tag{3}$$

it follows that

$$MM' < \varepsilon, \tag{4}$$

which will lead to a contradiction with b).

Let $\varepsilon > 0$ be given. Choose $\varepsilon_1 > 0, \varepsilon_2 > 0$ so that

$$\varepsilon = 2[R\varepsilon_1 + (1 + 1/k)\varepsilon_2(1 + \varepsilon_1)], \tag{4_1}$$

where $R = OM$; then choose n_1, n_2 so that for all $n > n_1,\ m \geq 1$ the requirement

$$(\widehat{\mathbf{l}_{i_n}, \mathbf{l}_{i_{n+m}}}) < \varepsilon_1, \tag{4_2}$$

is satisfied, and for all $n > n_2,\ m \geq 1$ the requirement

$$OO_{i_n} < \varepsilon_2, \qquad OO_{i_{n+m}} < \varepsilon_2 \tag{4_3}$$

is satisfied.

Choose an arbitrary $n > \max(n_1, n_2)$; we shall prove that it is the desired one. For this purpose introduce auxiliary screw motions L^* and L^{**} (they need not belong to the given group): $L^* \sim O, k, \mathbf{l}_{i_n}, \varphi$; $L^{**} \sim O, k, \mathbf{l}_{i_{n+m}}, \varphi$.

Now let $M^* = (L^*)^{-1} L_{i_n}(M)$, and $M^{**} = (L^{**})^{-1} L^*(M^*)$, if M' is chosen according to (3), then $M' = L_{i_{n+m}}^{-1} L^{**}(M^{**})$. Then, by Lemma 1 and condition (4_2), $M^*M^{**} < 2 \cdot OM^* \cdot \varepsilon_1$; by Lemma 2 and condition (4_3):

$$MM^* < (1 + 1/k)\varepsilon_2, \qquad M^{**}M' < (1 + 1/k)\varepsilon_2; \tag{4_4}$$

hence $OM^* < R + (1 + 1/k)\varepsilon_2$, and therefore

$$M^*M^{**} < 2[R + (1 + 1/k)\varepsilon_2]\varepsilon_1. \tag{4_5}$$

Applying the triangle inequality and equality (4_1), from conditions (4_4) and (4_5) we obtain (4), i.e. the chosen n is indeed the desired one.

Since the point M was chosen arbitrarily, requirement b) is violated. For the case $P = L$ the theorem is proved.

The remaining three cases present no difficulty. The case $P = K$ may be regarded as particular with respect to the basic one ($\varphi = 0$, $\mathbf{l}$ is chosen arbitrarily), while in fact the reasoning can be considerably simplified. The case $P = M$ reduces to the preceding one, and $P = \overline{L}$ to the basic one by replacing P by P^2. The theorem is completely proved.

4. As is not hard to see, the basic idea of the proof is suitable for any n (but for $n = 2$ there is no need to extract a subsequence). In a space of dimension $n > 3$ the matrix of any similarity transformation with coefficient $k \neq 1$ can (by choosing an orthonormal frame) be brought to the form $k \cdot A$, where the matrix A either has diagonal form with the numbers ± 1 on the main diagonal, or block-diagonal form, where m blocks $A_1, \dots, A_m$ are matrices of the form

$$A_i = \begin{pmatrix} \cos\varphi_i & -\sin\varphi_i \\ \sin\varphi_i & \cos\varphi_i \end{pmatrix} \qquad (i = 1, \dots, m),$$

and the remaining $n-2m$ diagonal elements are likewise equal to ± 1. Replacing P by P^2, one can eliminate the minus signs before the units. The basic form of P for the proof will be the "screw motion" $L \sim O, k, (l_1, \dots, l_m), (\varphi_1, \dots, \varphi_m)$, where $l_1, \dots, l_m$ are bivectors characterizing the "axis" of a compound rotation through the angles $\varphi_1, \dots, \varphi_m$ about a system of m distinct $(n-2)$-dimensional planes; all considerations connected with removing singularities of a point, constructing the sequence $L_1, L_2, \dots$, and applying the Bolzano–Weierstrass principle remain valid.

It is not fundamentally difficult to construct, as was done for $n = 2$ (^{4}), such a homeomorphic mapping of the hypersurface of the "straight spherical cylinder" in $(n+1)$-dimensional space* onto a hyperplane with one point removed; this naturally entails an isomorphic mapping of the symmetry group of the "directed" cylindrical hypersurface (i.e., with mutually nonequivalent opposite directions along the generators) onto the n-dimensional group of similarity transformations preserving the "puncture" of the hyperplane. It is enough to generalize directly the considerations of p. 3 of (^{4}). Consequently, it is expedient to investigate in parallel the n-dimensional symmetry groups of similarity and the $(n+1)$-dimensional linear (^{8}) groups of ordinary symmetry, and to transfer to both the idea of antisymmetry, etc.

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* The generator of the cylinder is a straight line, and the directrix is an $(n-1)$-dimensional surface of a sphere in a hyperplane perpendicular to the generator.

Note: Figure translations are in progress. See original paper for figures.

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