ASYMPTOTIC GEODESICS ON A RIEMANNIAN MANIFOLD NON-HOMEOMORPHIC TO THE SPHERE

MATHEMATICS

1966

SovietRxiv

Abstract

Full Text

UDC 513.573.3

MATHEMATICS

E. V. GAIDUKOV

ASYMPTOTIC GEODESICS ON A RIEMANNIAN MANIFOLD NON-HOMEOMORPHIC TO THE SPHERE

(Presented by Academician A. N. Kolmogorov on 25 XI 1965)

§ 1. **Formulation of the result.** 1. As the simplest application of Theorem 1 proved below, let us consider the plane double pendulum. Its position is determined by two angular coordinates φ and ψ; the configuration space of the double pendulum is the two-dimensional torus. Applying Theorem 1 to the case of the two-dimensional torus, we obtain the following result:

Whatever the initial values φ_0 and ψ_0 of the coordinates φ and ψ may be, one can choose such initial velocities $\dot{\varphi}_0$ and $\dot{\psi}_0$ that the motion of the pendulum determined by the initial conditions $((\varphi_0, \psi_0, \dot{\varphi}_0, \dot{\psi}_0)$ will asymptotically approach such a periodic motion in which one of the links of the pendulum makes k revolutions and the other l revolutions during one and the same fixed interval of time (k and l are arbitrarily prescribed integers).

An analogous assertion is valid for any dynamical system with two degrees of freedom.

2. Everywhere in what follows, the letter M denotes a smooth oriented two-dimensional Riemannian manifold, complete in its metric ([1]). All curves on M are assumed to be piecewise smooth. Two curves are said to be **freely homotopic** on the manifold M if they can be continuously deformed into one another without leaving the manifold M during the deformation (for the precise definition see ([2])).

Theorem 1. *Let x be an arbitrary point of a compact manifold M not homeomorphic to the two-dimensional sphere; let γ be an arbitrary class of freely homotopic closed paths of the manifold M. Then there exists a half-geodesic L issuing from the point x, asymptotic to some closed geodesic Π of the class γ.*

The proof of this theorem is based on simple geometric arguments set forth in the following paragraphs.

§ 2. **Minimal curves.** In this paragraph we shall formulate several well-known propositions concerning properties of minimal curves which will be used below.

Fig. 1

Figure 1: Fig. 1

1. A curve Γ on the manifold M is called **minimal between two points** $x, y \in \Gamma$ if the segment $\Gamma(x, y)$ of the curve Γ enclosed between the points x and y has length not exceeding the length of any curve on M joining the points x and y. A **minimal curve** is a curve that is minimal between any two of its points.
2. Let Λ_1 and Λ_2 be two minimal curves with endpoints x_1, y_1 and x_2, y_2, respectively. Then only the following cases are possible:

a) the curves Λ_1 and Λ_2 intersect in no more than one point;

b) the curves Λ_1 and Λ_2 are continuations of one another; in this case their intersection is nonempty and is a minimal curve (this intersection may consist of one point);

c) the curves Λ_1 and Λ_2 coincide;

d) the curves Λ_1 and Λ_2 intersect only at their common endpoints.

3. From item 2 there follows an assertion important for what follows:

Lemma 1. Two minimal curves with a common initial point at a point $x \in M$ intersect in no more than one point distinct from x, except for the case in which one of these curves is entirely contained in the other.

§ 3. **Minimal geodesics on a surface of genus > 0.** Beginning with this paragraph, the letter M denotes a compact surface not homeomorphic to the sphere.

1. The universal covering manifold of M is the two-dimensional plane, endowed with the induced metric; the projection $p : \widetilde{M} \to M$ is locally isometric. The following assertions are valid:

a) If Λ is a minimal arc on M, then the covering path $\widetilde{\Lambda}$ is minimal on $\widetilde{M}$.

Fig. 1

b) Let Γ be a minimal geodesic of the class γ of freely homotopic paths of the manifold M. Then the curve Γ is a minimal loop at each of its points; the preimage of each such loop is a minimal arc on $\widetilde{M}$ with endpoints $\tilde{y}_1$ and $\tilde{y}_2$: $p(\tilde{y}_1) = p(\tilde{y}_2)$.

2. From 1a) it follows directly:

The curve $\tilde{\Gamma}$, lying over the minimal closed geodesic $\Gamma \in \gamma$, is a geodesic on $\widetilde{M}$.

3. **Lemma 2.** The geodesic $\tilde{\Gamma}$ of item 2 is minimal between any two of its points.

We preface the proof of the lemma with the following simple observation. The fundamental group $\pi_1(M)$ of the manifold M is naturally isomorphic to the group of all motions of the universal covering $\widetilde{M}$ lying over the identity motion of the manifold M. By virtue of this isomorphism we identify motions of the covering with elements of the group π^1 and denote them by the same letters.

4. **Proof of Lemma 2.** Let $\Gamma \in \gamma$ be a minimal geodesic and let $\tilde{\Gamma}$ lie over Γ. Let $\tilde{y}$ be an arbitrary point on the geodesic $\tilde{\Gamma}$. Put

$$\tilde{y}_k = \gamma^k(\tilde{y}), \quad k = 0, \pm 1, \pm 2, ...$$

(Fig. 1).

Each of the segments $\tilde{\Gamma}(\tilde{y}_k, \tilde{y}_l)$ of the geodesic $\tilde{\Gamma}$, enclosed between the points $\tilde{y}_k$ and $\tilde{y}_l$, is minimal (1b). We shall prove that the segment $\tilde{\Gamma}(\tilde{y}, \tilde{y}_2)$ is minimal. If this is not so, then there exists a minimal arc S joining the points $\tilde{y}$ and $\tilde{y}_2$, having length strictly less than the length of the arc $\tilde{\Gamma}(\tilde{y}, \tilde{y}_2)$. Put $S_1 = \gamma(S)$, $S_{-1} = \gamma^{-1}(S)$ (Fig. 1). The three minimal curves S, S_1, and S_{-1} obviously intersect at certain points $\tilde{u}$ and $\tilde{v}$ (see Fig. 1).

Denote the length of the arc $\tilde{\Gamma}(\tilde{y}, \tilde{y}_1)$ by $|\gamma|$, and the distance function by ρ. We have

$$\rho(\tilde{y}, \tilde{u}) + \rho(\tilde{v}, \tilde{y}_2) \geq |\gamma|.$$

At the same time:

a) $\rho(\tilde{u}, \tilde{v}) \geq |\gamma|$, for Γ is minimal in its class of free homotopy;

b) the length $(S) = \rho(\tilde{y}, \tilde{u}) + \rho(\tilde{u}, \tilde{v}) + \rho(\tilde{v}, \tilde{y}_2) \geq 2|\gamma|$, i.e., the length of the arc S is not less than $2|\gamma|$, contrary to the supposition. The proof of the lemma is now easily completed by induction.

§ 4. Construction of an asymptotic ray on the covering.

1. Let $\tilde{y}$ and $\tilde{y}'$ be two distinct points lying over some point $y \in \Gamma$. Let, furthermore, $\tilde{\Gamma}$ and $\tilde{\Gamma}'$ be minimal geodesics passing through the points $\tilde{y}$ and $\tilde{y}'$, respectively, and covering the minimal closed geodesic Γ of the class γ.

Let T be some minimal arc joining the points $\tilde{y}$ and $\tilde{y}'$. Put $T_k = \gamma^k(T)$, $k = 0, \pm 1, \pm 2, ...$ (Fig. 2).

2. Choose a point $\tilde{x}$ over the given point $x \in M$, lying in the strip Q between the geodesics $\tilde{\Gamma}$ and $\tilde{\Gamma}'$. Let $\widetilde{\Lambda}_k$, $k = 0, 1, 2, ...$, be some minimal arcs joining the point $\tilde{x}$ to the points $\tilde{y}_k$. Denote by ξ_k the linear element of the arc $\widetilde{\Lambda}_k$ at the point $\tilde{x}$. An immediate consequence of Lemma 1 is the following assertion:

Fig. 2

Figure 2: Fig. 2

The vector ξ_k rotates monotonically; consequently, there exists a limiting direction:

$$\xi_\infty = \lim_{k\to+\infty} \xi_k.$$

3. Let $\widetilde{L}$ be the half-geodesic issuing from the point $\tilde{x}$ in the direction ξ_∞. From the properties of minimal curves (§ 2) it follows that:

a) *The half-geodesic $\widetilde{L}$ is minimal between any two of its points, i.e. it is a geodesic ray* ([3]).

b) *The geodesic ray $\widetilde{L}$ does not intersect the curve* Γ.

4. Let α_k be the coordinate of the point of intersection of the curve $\widetilde{L}$ with the arc T_k, measured from the point $\tilde{y}_k$ along the arc T_k (we take the coordinate of the point $\tilde{y}_k$ to be zero, and the length of the arc T_k to be one).

Lemma 3. *The sequence α_k is monotone.*

It is enough to establish the monotonicity of the corresponding sequence for each of the curves $\widetilde{\Lambda}_i$. This is easily done with the aid of arguments analogous to those given in § 3, item 4.

Fig. 2

5. Consider the sequence of iterations $\widetilde{L}^{-n} = \gamma^{-n}(\widetilde{L})$ of the geodesic ray $\widetilde{L}$. From the preceding lemma the following follows.

Main assertion. *The sequence of geodesic rays $\widetilde{L}^{-n}$ converges to a certain geodesic $\widetilde{\Pi}$ on the manifold $\widetilde{M}$. The geodesic $\widetilde{\Pi}$ is minimal between any two of its points and is invariant with respect to the motion γ. Moreover, the geodesic ray $\widetilde{L}$ is asymptotic to the geodesic* $\widetilde{\Pi}$.

6. Projecting the geodesic $\widetilde{\Pi}$ and the ray $\widetilde{L}$ onto the manifold M, we obtain a closed geodesic Π of the class γ and a half-geodesic L, asymptotic to Π. Theorem 1 is proved.

7. **Remark.** Generally speaking, the limiting geodesic Π need not coincide with the initial closed geodesic Γ. Nor is it necessarily minimal in the class γ. However, the following assertion is true:

The geodesic Π and the asymptote L do not contain conjugate points.

Moscow State University
named after M. V. Lomonosov

Received
17.XI.1965

REFERENCES

1. H. Hopf, W. Rinow, *Comm. math. Helv.*, **3**, 209 (1931).

2. G. Seifert, W. Threlfall, *Topology*, Moscow–Leningrad, 1938.

3. S. E. Cohn-Vossen, *Shortest paths and problems of curvature*, in the book: S. E. Cohn-Vossen, *Some problems of differential geometry in the large*, Moscow, 1959.

Note: Figure translations are in progress. See original paper for figures.