A CRITERION OF FUNCTIONAL COMPLETENESS IN THE LOGIC CORRESPONDING TO JAŚKOWSKI’ S FIRST MATRIX

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A CRITERION OF FUNCTIONAL COMPLETENESS IN THE LOGIC CORRESPONDING TO JAŚKOWSKI' S FIRST MATRIX

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1. The simplest of the propositional logics intermediate between classical logic and constructive (i.e., intuitionistic) logic is, as is well known, the logic corresponding to the so-called first Jaśkowski matrix*. This logic, called (at the suggestion of A. A. Markov) Smetanich' s logic**, is simple in the sense that it is constructed tabularly, as a variant of three-valued logic in which the basic operations—conjunction &, disjunction ∨, implication ⊃, and negation ¬—are given by the tables of Jaśkowski' s first matrix. Smetanich' s logic is intermediate between constructive logic and classical logic not only in the degree of simplicity of its construction, but also in the sense that every logical relation having the form of an equivalence (identity) of formulas, implication, etc., if true in constructive logic, is true also in Smetanich' s logic, and if true in the latter, is true also in classical logic (but not conversely). In this connection, it is natural to try first to solve, for Smetanich' s logic, certain questions that have been solved for classical logic but still present considerable difficulties for constructive logic. Examples of questions of this kind are those of functional completeness, an attempt at a systematic consideration of which for constructive logic was undertaken by A. V. Kuznetsov (^{5}).

The consideration of questions of functional completeness for Smetanich' s logic begins with the consideration of the notion of a function of Smetanich' s logic as a function that can be expressed by a formula of propositional logic whose operation signs are interpreted according to the tables of Jaśkowski's first matrix, i.e., in other words, expressed through the basic operations of this logic. Under such a consideration, there arises first of all the question of the relation of this notion to the general notion of a function of three-valued logic (a function from P_3), since questions of functional completeness for the usual variant of three-valued logic (when the class of all functions that can be specified by tables with three values is considered) were investigated by S. V. Yablonskii (^{6,7}). Such a consideration may also be connected with the problem of a more detailed investigation than before of classes of functions of three-valued logic—a problem that naturally arises in connection with the results of Yu. I. Yanov and A. A.

Muchnik (^{8}), who discovered an essential difference between three-valued logic and two-valued (i.e., classical) logic.

2. We shall denote propositional **variables** by the letters x, y, z, and u, possibly with subscripts. The **values** of variables are denoted by $0, \tau, 1$*. The class of all such functions, the domain of values of each of whose arguments is the set $\{0, \tau, 1\}$ and whose values belong to the same set, is denoted (see (7)) by the symbol P_3. In what follows, wherever the contrary is not specified, by the term **function** we shall mean a function from P_3.

We define the operations $\&, \vee, \supset$, and $\neg$ by tables, the totality of which (with the indication that the designated value, i.e. truth, is taken to be 1) is called **Jaśkowski' s first matrix**:

$(x\&y)$

$x\backslash y$	0	τ	1
0	0	0	0
τ	0	τ	τ
1	0	τ	1

$(x \vee y)$

$x\backslash y$	0	τ	1
0	0	τ	1
τ	τ	τ	1

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* On Jaśkowski matrices see (^{1,2}). The corresponding structures, called Jaśkowski structures, were considered by V. A. Yankov (^{3}).

** Ya. S. Smetanich, in an article (^{4}) and in other works, was apparently the first to study this logic not only in connection with other logics, but also by singling it out for more special consideration. He also, as V. A. Yankov reports in (^{3}), proposed constructing a calculus corresponding to this logic by adding to the constructive (intuitionistic) propositional calculus a new axiom

$$((x \supset y) \vee (y \supset z)) \vee (z \supset u)).$$

A. V. Kuznetsov (^{5}) defines Smetanich' s logic as the logic of a three-element structure.

$x\backslash y$	0	τ	1
1	1	1	1

$$(x \supset y)$$

$x\backslash y$	0	τ	1
0	1	1	1
τ	0	1	1
1	0	τ	1

$$\neg x$$

x	$\neg x$
0	1
τ	0
1	0

A function (from P_3) is called a **function of Smetanich logic** if it is expressible (by means of superpositions) through the functions

$$(x \& y);\ (x \vee y),\ (x \supset y),\ \neg x. \qquad (\Sigma_0)$$

We denote the class of all functions of Smetanich logic by the letter S. An example of a function not belonging to the system Σ_0 but belonging to the class S is **equivalence** $(x \sim y) = ((x \supset y)\ \&\ (y \supset x))$.

We say that a function $f(x_1, \dots, x_m)$ **preserves** a predicate $P(x_1, \dots, x_n)$ if, for all possible values of the variables x_{ij} $(i = 1, \dots, m;\ j = 1, \dots, n)$, from the fact that $P(x_{11}, x_{12}, \dots, x_{1n})$, $P(x_{21}, x_{22}, \dots, x_{2n}), \dots, P(x_{m1}, x_{m2}, \dots, x_{mn})$ holds, it follows that $P(f(x_{11}, x_{21}, \dots, x_{m1}), \dots, f(x_{1n}, x_{2n}, \dots, x_{mn}))$ holds. The set of all functions preserving the predicate P is called the **preservation class** of the predicate P (see (9), pp. 104-105, and also (10)).

3. We denote the preservation class of the predicate $x \neq \tau$ by the letter T, and the preservation class of the predicate $\neg x = \neg y$ by the letter U**. Notice that the constant τ belongs to the class U, but does not belong to T. An example of a function belonging to T but not to U is the function $\rfloor x$ (weak negation), defined by the equalities $\rfloor 0 = 1$, $\rfloor \tau = 1$, $\rfloor 1 = 0$ (cf. (11)). From consideration of the tables given above it is clear that all functions

of the system Σ_0 belong to the class $T \cap U$. It is also not difficult to show that to the latter belongs every such and only such function $f(x_1, \dots, x_n)$ for which the identity

$$\neg\neg f(x_1, \dots, x_n) = f(\neg\neg x_1, \dots, \neg\neg x_n) \tag{1}$$

holds.

Theorem 1. *$S = T \cap U$.*

Theorem 1′. In order that a function $f(x_1, \dots, x_n)$ belong to the class S, it is necessary and sufficient that it satisfy the identity (1).

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[*] The sign 0 is convenient for denoting falsehood, the sign 1 for denoting truth, especially when simultaneously considering classical, constructive, and intermediate logics, also taking into account the traditions in the algebra of logic going back to Boole and Żegalkin; the sign τ is convenient for denoting a third value, intermediate between falsehood and truth. S. V. Yablonskii and a number of other authors denote the variables differently; however, for the convenience of using their constructions and results, in the present work this difference in notation is regarded as inessential and is ignored in what follows.

[**] This notation for these two classes was chosen because the first of them is one of the classes called in (7) classes of type T (namely, the class of functions preserving the set $\{0, 1\}$), and the second of them is one of the classes called in (7) classes of type U (the class of functions preserving the partition $\{\{0\}, \{\tau, 1\}\}$).

To prove Theorems 1 and 1′, in view of what was said above, it is enough to prove that every function f belonging to $T \cap U$ is expressible in terms of the functions of the system Σ_0. The proof of the latter is based on the fact that every such function f is either the constant $0 = (x \& \neg x)$, or a disjunction of certain functions g, for each of which the number of such tuples (of argument values) not containing τ on which g differs from 0 is equal to one; and each such function g is equal to the conjunction of certain functions of the form $\neg x_i$, or of the form $\neg\neg x_j$, or of the form $(A \supset B)$, or of the form B, where A is a conjunction of certain variables, and B is a disjunction of certain variables.

4. A system (class) Σ of functions from a given class K will be called **(functionally) complete in** K if all functions of the class K are expressible in terms of functions from Σ. A system Σ of functions from the class K will be called **precomplete in** K if Σ is not complete in K, but for every function f from K not belonging to Σ, the system $\Sigma \cup \{f\}$ is complete in K.

Theorem 2. *The class S is precomplete in the class T.*

For the proof it is enough to show the following: 1) if $f \in T$ and $f \notin S$, then the function $\neg x$ is expressible in terms of f and functions from S; 2) the system

$\Sigma_0 \cup \{\neg x\}$ is complete in T.

Theorem 3. *The class S is precomplete in the class U.*

It is enough to show the following: 1) if $f \in U$ and $f \notin S$, then the constant τ is expressible in terms of f and functions from S; 2) the system $\Sigma_0 \cup \{\tau\}$ is complete in U.

Let us note that the classes T and U themselves, as is clear from their definition (taking also into account the footnote to it) and from work (^{7}), are precomplete in P_3.

For each of the following 10 predicates we shall consider the class of all functions from S preserving the given predicate

$$x = 0, \qquad x = 1, \qquad \neg x \neq \neg y, \qquad \neg x \leqslant \neg y,$$

$$\neg(x \sim y) = \neg(z \sim u), \qquad (x \& y) \neq \tau, \qquad ((x \vee y) \vee (x \sim y)) = 1,$$

$$((x \subset y) \& (\neg x \supset \neg y) = 1, \qquad ((x \vee \neg x \vee y) \& (\neg x \sim \neg y)) = 1, \qquad (\,)$$

$$\neg\neg((x \& y) \vee (x \& z) \vee (y \& z)) = (x \sim (y \sim z));$$

the ten classes thus defined will be denoted respectively by the symbols $S_0, S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8, S_9$.*

For comparing the 10 classes just defined with the known 18 classes precomplete in P_3 (^{7}), the following two theorems are useful.

Theorem 4. *If the class K coincides either with S_0, or with S_1, or with S_6, then there exists a class J precomplete in P_3 such that $K = J \cap S$.*

Theorem 5. *For every class J precomplete in P_3, the intersection $J \cap S$ is equal to the class S, or equal to S_0, or equal to S_1, or equal to S_6, or is strictly contained in at least one of the classes S_0, S_1, S_3, S_5, S_6.*

5. **Theorem 6 (criterion of functional completeness in Smetanich' s logic).** *In order that a system Σ of functions from the class S be complete in S, it is necessary and sufficient that for each $i = 0, 1, \ldots, 9$ there exist a function belonging to Σ but not belonging to S_i.*

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* Cf. the first 5 predicates of the list ( ) with the 5 predicates (see (^{9}), p. 105), whose preservation classes are precomplete in P_2 (i.e. in two-valued logic). The role played by the last three of them is, to some extent, analogous to the role of the classes S_0, S_1, S_2, S_3, S_4 in S. This is seen from comparing Lemmas 1, 2, and

3 with the proof of sufficiency of the criterion of functional completeness in P_2 (see (^{7}), pp. 18-20), and is explained by the existence of such a homomorphism (in the sense of § 8 of work (^{7})) of the class S onto the class P_2, under which the images of the classes S_0, S_1, S_2, S_3, S_4 are classes precomplete in P_2. As for the remaining 5 predicates of the list ( ), attention should be paid to the following property of the last three of them, namely that the predicate under consideration entails the predicate $\neg x = \neg y$, occurring in the definition of the class U. Cf. the first two of these three predicates with the formulas used for the proof of Lemma 2 of work (^{5}). The last of these two predicates is more simply written as:

$$\neg\neg(x \& y) = (x \vee y).$$

The necessity of the criterion follows from the fact that the classes $S_0, S_1, \dots, S_9$ are closed with respect to superpositions and are strictly contained in S. The proof of sufficiency is based on the fact that, denoting by the symbols $f_0, f_1, \dots, f_9$ functions (from Σ) that do not belong respectively to $S_0, S_1, \dots, S_9$, one can prove the following lemmas.

Lemma 1. Through the functions f_0, f_1, and f_2 the constants 0 and 1 are expressible.

Lemma 2. Through the functions $0, 1$, and f_3 the function $\neg x$ is expressible.

Lemma 3. Through the functions $0, 1, \neg x$, and f_4 every function of the class S not taking the value τ is expressible.

Lemma 4. Through the functions f_5, f_6, f_7, f_8, f_9 and functions of the class S not taking the value τ, the functions $(x \& y)$ and $(x \supset y)$ are expressible.

Lemma 5.

$$(x \vee y) = (((x \supset y) \supset y) \& ((y \supset x) \supset x)).$$

6. The subsequent theorems are, in essence, consequences of Theorem 6.

Theorem 7. There exist exactly 10 precomplete classes of functions in S, namely the classes $S_0, S_1, \dots, S_9$.

We call a function $f(x_1, \dots, x_n)$ **trivial** if there exists a variable x_i such that $f(x_1, \dots, x_n) = x_i$ for all values of the variables; otherwise we call it nontrivial.

Theorem 8. There exists a nontrivial function belonging simultaneously to all precomplete classes in S.

For the proof it suffices to give an example: the function $\neg\neg x$ or the function $(x \vee \neg(x \supset y))$.

Theorem 9. In every complete system of functions in S there exists a subsystem, complete in S, containing no more than 9 functions.

We call a function of a class K **Sheffer in** K if all functions of the class K are expressible through it (cf. (^{7}), p. 77, and also (^{5})).

Theorem 10. In order that a function of the class S be Sheffer in S, it is necessary and sufficient that it belong to none of the following six classes: $S_0, S_1, S_2, S_7, S_8, S_9$. A system Σ of functions from K is called **weakly complete** in K if the system consisting of the functions belonging to Σ and the constants belonging to K is complete in K (cf. ([9]), p. 104).

Theorem 11. In order that a system Σ of functions of the class S be weakly complete in S, it is necessary and sufficient that for each $i = 3, 4, \ldots, 9$ there exist a function belonging to Σ but not belonging to S_i.

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Note: Figure translations are in progress. See original paper for figures.

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