INTEGRAL REPRESENTATIONS OF HOLOMORPHIC FUNCTIONS OF SEVERAL COMPLEX VARIABLES

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INTEGRAL REPRESENTATIONS OF HOLOMORPHIC FUNCTIONS OF SEVERAL COMPLEX VARIABLES

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1. As is known, in the classical theory of holomorphic functions of one complex variable z, the Cauchy formula is of fundamental importance. One of its remarkable features is that the denominator of the integrand, with respect to the internal variable z, is a polynomial of the first degree. In the case of n $(n > 1)$ complex variables $z_1, \ldots, z_n$, the Cauchy formula loses this feature, since the denominator of its kernel is a polynomial of the n-th degree in the variables $z_1, \ldots, z_n$. At present, besides the Cauchy formula $(n > 1)$, many integral representations are known ([1–5]) which extend the Cauchy formula $(n = 1)$ to the case of holomorphic functions of n $(n > 1)$ complex variables; however, they likewise do not possess the indicated feature of the Cauchy formula $(n = 1)$. In this connection one should note the Temlyakov integral representation of the first kind ([1,6–8])–the only integral representation in the case $n > 1$ which not only preserves the above-mentioned feature of the Cauchy formula $(n = 1)$, but also has as its kernel the Cauchy kernel $(n = 1)$. This first distinctive property of the Temlyakov integral representation of the first kind is accompanied, along with a number of circumstances, by the following second property of this integral representation, which substantially distinguishes it from the integral representations mentioned above in the case $n > 1$. It expresses the value of a function $F(z_1, \ldots, z_n)$ inside a domain not through the values of this function on the boundary (or on part of the boundary) of the domain, but through the values on the boundary (or on a part of it) of a linear differential operator of order $n - 1$ applied to F. The two noted properties of the Temlyakov integral representation of the first kind are inherent in holomorphic functions only in the case $n > 1$, for in passing to $n = 1$ one or the other of these properties disappears. The Temlyakov integral representation of the first kind was found ([1,6–8]) for convex complete polydisc domains. In the present note, for these same domains of the space C^n of n $(n > 1)$ complex variables, an integral formula is obtained ((3), $\alpha = 1$, $k = \mu = n - 2$) with the above-indicated feature of

the Cauchy formula ($n = 1$). In this formula there enter Cauchy integrals ($n = 1$)*. The latter formula is a consequence of a general integral representation for functions holomorphic in bounded convex complete n-circular domains (Theorem 2), including $3 \cdot 2^{n-2}$ ($n \geq 2$) integral representations, two of which for $n = 2$ ($\alpha = 0$) had been obtained much earlier by A. A. Temlyakov ([6,9,10]), and, for $n > 2$ ($\alpha = 0,\ m_1 = 1, \dots, m_{n-1} = n-1$), in another form by the Polish mathematicians Opial and Siciak ([8]), while all the others are new. The corresponding theorem (Theorem 3) is also stated for the case of convex domains of the space C^n ($n \geq 2$). Here the integral formula, which in view of brevity of exposition is not written out, likewise includes $3 \cdot 2^{n-2}$ integral representations, one of which for $\alpha = \mu = 0$ had earlier been obtained—

* As the last inner integrals.

by L. A. Aizenberg ([4]), and all the remaining ones are new. In the present note we establish also certain integral formulas of a reconstructive character (Theorem 1), in which an auxiliary role is played by functions holomorphic in star-shaped domains of the space C^n, $n \geqslant 1$.

2. Let G be a domain star-shaped* with respect to the origin in the space C^n of n complex variables $z_1, \dots, z_n$, $n \geqslant 1$; let $F(z_1, \dots, z_n)$ be a function holomorphic in the domain G;

$$L_m[F(z_1, \dots, z_n)] \equiv$$

$$\equiv mF(z_1, \dots, z_n) + \sum_{k=1}^{n} z_k F'_{z_k}(z_1, \dots, z_n) \quad (m = 0, 1, 2, \dots)$$

be the operator introduced by A. A. Temlyakov ([11]). Let, further, p, q be natural numbers with $p \geqslant q$, and let $m_p, m_{p-1}, \dots, m_q, m_0$ be arbitrary natural numbers. We introduce the notation (for brevity, instead of $F(z_1, \dots, z_n)$ we write F)

$$L^{(p-q+1)}_{\binom{m_q}{m_p}}[F] \equiv L_{m_p}[L_{m_{p-1}} \cdots [L_{m_q}[F]] \cdots]$$

and put

$$L^{(0)}_{\binom{m_p}{m_{p-1}}}[F] \equiv F \qquad (p \geqslant 1).$$

3. Suppose that τ is a real variable ranging over the segment $0 \leqslant \tau \leqslant 1$.

Theorem 1. If the function $F(z_1, \dots, z_n)$ ($n \geqslant 1$) is holomorphic in the domain G, then in G the formulas

$$F(z_1, \dots, z_n) = \int_0^1 \tau^{m-1} L_m[F(\tau z_1, \dots, \tau z_n)]\, d\tau, \qquad m = 1, 2, \dots, \tag{1}$$

$$F(z_1, \dots, z_n) = F(0, \dots, 0) + \sum_{k=1}^{n} z_k \int_0^1 F'_{\tau z_k}(\tau z_1, \dots, \tau z_n)\, d\tau. \tag{2}$$

hold.

Formula (1) reconstructs at each point $(z_1, \dots, z_n) \in G$ the function F from its operator $L_m[F]$ on the segment joining the point $(z_1, \dots, z_n)$ to the origin. For $m = 1$ this is a new property of the operator $L_1[F]$, in addition to its known property obtained by A. A. Temlyakov ([6]). Formula (2), up to the constant term $F(0, \dots, 0)$, reconstructs at each point $(z_1, \dots, z_n) \in G$ the function F from all its first partial derivatives taken on the segment joining the point $(z_1, \dots, z_n)$ to the origin.

Proof. Let $(z_1, \dots, z_n)$ be an arbitrary point of the domain G. Since the domain G is star-shaped with respect to the origin, for $0 \leqslant \tau \leqslant 1$ the points $(\tau z_1, \dots, \tau z_n) \in G$. Now, for the validity of formulas (1), (2), it suffices to note that

$$\tau^{m-1} L_m[F(\tau z_1, \dots, \tau z_n)] = (\tau^m F(\tau z_1, \dots, \tau z_n))'_\tau,$$

$$\sum_{k=1}^{n} z_k F'_{\tau z_k}(\tau z_1, \dots, \tau z_n) = F'_\tau(\tau z_1, \dots, \tau z_n).$$

4. In ([8]) the definition of a domain of type (T) was introduced and two propositions were proved.

Let Δ be an $(n-1)$-dimensional simplex,

$$\Delta = \{\tau = (\tau_1, \dots, \tau_n) : \ \tau_1 + \dots + \tau_n = 1,\ \tau_1 > 0, \dots, \tau_n > 0\}.$$

We shall say that a bounded domain D in the space C^n of complex variables $z = (z_1, \dots, z_n)$, $n \geqslant 2$, is a domain of type (T) (briefly, $D \in (T)$) if there exist positive real functions

$$r_k = r_k(\tau), \qquad k = 1, \dots, n, \tag{A}$$

* A domain is called star-shaped with respect to the origin if, together with each point, it contains the entire segment joining this point to the origin.

defined and continuous for $\tau \in \Delta$, such that

$$D = \bigcup_{\tau\in\Delta} \{z: \ |z_k| < r_k(\tau), \quad k = 1, \dots, n\}, \tag{B}$$

$$D = \text{int.} \bigcap_{\tau\in\Delta} \left\{ z: \ \sum_{k=1}^{n} \tau_k r_k^{-1}(\tau)|z_k| < 1 \right\}. \tag{C}$$

Put

$$\Delta^* = \{(\tau_2, \dots, \tau_n): \ 0 < \tau_2 < 1, \quad 0 < \tau_3 < 1 - \tau_2, \dots, \ 0 < \tau_n < 1 - \tau_2 - \dots - \tau_{n-1}\}.$$

I. If $D \in (T)$ and if a function $f(z)$ holomorphic in D and all its partial derivatives up to order μ $(0 \le \mu \le n-1)$ inclusive are continuous in $D \cup S$, where

$$S = \{z: \ z_k = r_k(\tau)\exp i\theta_k, \quad 0 \le \theta_k \le 2\pi, \quad k = 1, \dots, n\}$$

is a part of the boundary ∂D of the domain D, then for $k = 0, 1, \dots, \mu$ and $z \in D$

$$f(z) = (n-k-1)![(2\pi)^n i]^{-1} \int d\omega_\tau \int d\omega_\theta \int_{|\zeta|=1} \zeta^{n-k-1}(\zeta - u)^{k-n} F_k(\zeta, r, \theta)\, d\zeta.$$

Here

$$F_0(\zeta, r, \theta) = f(\zeta r_1, \zeta r_2 \exp i\theta_2, \dots, \zeta r_n \exp i\theta_n),$$

$$F_k = (n-k)F_{k-1} + \zeta F'_{k-1,\zeta}, \quad k = 1, \dots, n-1,$$

$$u = \tau_1 r_1^{-1}(\tau) z_1 + \sum_{l=2}^{n} \tau_l r_l^{-1}(\tau)\exp(-i\theta_l) z_l,$$

$$\int d\omega_\tau = \int_{\Delta^*} d\tau_2 \cdots d\tau_n, \qquad \int d\omega_\theta = \int_0^{2\pi} d\theta_2 \cdots \int_0^{2\pi} d\theta_n,$$

and the circle $|\zeta| = 1$ is oriented in the positive direction.

II. In order that a bounded domain D in C^n belong to type (T), it is necessary and sufficient that it be a convex complete n-circular domain. For every bounded convex complete n-circular domain D there exists a unique system of functions (A) satisfying conditions (B) and (C).

It turns out that the following general theorem 2 holds.

Theorem 2. Let $D \in (T)$, and let the function $f(z)$ $(n \ge 2)$ be holomorphic in D and assume the values 0 and 1. Then, if the functions $f^{(\alpha)}_{z^\alpha_\nu}(z)$, $\nu = 1, \dots, n$, and all their partial derivatives up to order μ $(0 \le \mu \le n-1-\alpha)$ inclusive are continuous in $D \cup S$, then for $k = 0, 1, \dots, \mu$ and $z \in D$

$$f(z) = \alpha f(0) + \frac{1}{n+\alpha(1-n)} \times$$
$$\times \sum_{\nu=1}^{n} \frac{z_\nu^\alpha}{(2\pi)^n i} \int d\omega_\tau \int d\omega_\theta \int_{|\zeta|=1} L_{(m_{\alpha+1})}^{(n-k-1-\alpha)} \left[\frac{1}{\zeta-u}\right] L_{(m_{n-1})}^{(k)} \left[F_{0\, z_\nu^\alpha}^{(\alpha)}(\zeta,r,\theta)\right] d\zeta, \tag{3}$$

where $m_{\alpha+1}, m_{\alpha+2}, \dots, m_{n-1}$ $(n \geq 3)$ are arbitrary, but all distinct natural numbers taken from $\{\alpha+1, \alpha+2, \dots, n-1\}$, and for $n=2$ and $\alpha=0$, $m_1=1$;

$$F'_{0z_\nu}(\zeta,r,\theta) = f'_\nu(\zeta r_1, \zeta r_2 \exp i\theta_2, \dots, \zeta r_n \exp i\theta_n)$$

$$(f'_\nu(z_1,\dots,z_n) = f'_{z_\nu}(z_1,\dots,z_n)), \qquad F_0^{(0)} = F_0.$$

In the proof one uses assertion I (in ([8]) this is theorem 1), formula (2), and the Cauchy formula in the case of one complex variable.

We shall call formula (3) the general Temlyakov integral representation. Let us note one important characteristic feature of this integral

representations. From formula (3) it is clear that the integral

$$\frac{1}{2\pi i} \int_{|\zeta|=1} L_{\binom{m_{\alpha+1}}{m_{n-k-1}}}^{(n-k-1-\alpha)} \left[\frac{1}{\zeta-u}\right] L_{\binom{m_{n-k}}{m_{n-1}}}^{(k)} \left[F_{z_\nu^\alpha}^{(\alpha)}(\zeta,r,\theta)\right] d\zeta$$

is

$$L_{\binom{m_{\alpha+1}}{m_{n-k-1}}}^{(n-k-1-\alpha)} \left[\frac{1}{2\pi i} \int_{|\zeta|=1} \frac{1}{\zeta-u} L_{\binom{m_{n-k}}{m_{n-1}}}^{(k)} \left[F_{z_\nu^\alpha}^{(\alpha)}(\zeta,r,\theta)\right] d\zeta\right],$$

i.e., either the Cauchy integral $(k = n-1-\alpha)$ in the case of one complex variable, or the given operator of this integral.

As already noted, formula (3) includes $3 \cdot 2^{n-2}$ $(n \geq 2)$ integral representations, of which $3 \cdot 2^{n-2} - n$ are new. Therefore, for $n=2$ we have one new integral representation, which we shall call the Temlyakov integral representation of the third kind.

5. Let the domain $D = \{(z_1,\dots,z_n) : \Phi(z_1,\bar{z}_1,\dots,z_n,\bar{z}_n) < 0\}$, containing the point $(0,\dots,0)$, be convex and bounded, and let the function Φ be twice continuously differentiable, with all first-order derivatives of Φ not vanishing simultaneously at points of the boundary ∂D of the domain D.

Theorem 3. Let the function $F(z_1, \dots, z_n)$ $(n \geq 2)$ be holomorphic in the domain D, and let α take the values 0 and 1. Then, if the functions $F^{(\alpha)}_{z^\alpha_\nu}(z_1, \dots, z_n)$ $(F^{(0)} = F)$, $\nu = 1, \dots, n$, and all their partial derivatives up to order μ $(0 \leq \mu \leq n - 1 - \alpha)$, inclusive, are continuous in the closed domain $\bar{D}$, then for $k = 0, 1, \dots, \mu$ and points $(z_1, \dots, z_n) \in D$ there holds a formula (in view of the brevity of the exposition we do not write it down here), analogous in character to formula (3), expressing the values of the function $F(z_1, \dots, z_n)$ in the domain D through the values of

$$L^{(k)}_{\binom{m_{n-k}}{m_{n-1}}}\left[F^{(\alpha)}_{z^\alpha_\nu}\right],$$

$\nu = 1, 2, \dots, n$, on the boundary ∂D, up to the addend $\alpha F(0, \dots, 0)$.*

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* In particular, for $\alpha = 1$ and $k = 0$—through the values of all first partial derivatives of F on the boundary ∂D, up to the addend $F(0, \dots, 0)$. Similarly in Theorem 2, but not on ∂D, rather on S.

Note: Figure translations are in progress. See original paper for figures.