ON SELF-SIMILARLY STABLE SOLUTIONS OF THE TRAVELING-WAVE TYPE FOR CERTAIN QUASILINEAR EQUATIONS, INCLUDING THE EQUATIONS OF WATER FLOW IN AN INCLINED CHANNEL

HYDROMECHANICS

1966

SovietRxiv

Fig. 1

Figure 1: Fig. 1

Abstract

Full Text

UDC 517.9:533.7

HYDROMECHANICS

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ON SELF-SIMILARLY STABLE SOLUTIONS OF THE TRAVELING-WAVE TYPE FOR CERTAIN QUASILINEAR EQUATIONS, INCLUDING THE EQUATIONS OF WATER FLOW IN AN INCLINED CHANNEL

(Presented by Academician V. I. Smirnov on 22 XI 1965)

I. Consider the quasilinear equation

$$u_t + uu_x = F(u). \tag{1}$$

We shall assume that $F(u)$ is a smooth function and that on any finite interval of variation of u, $F(u)$ has no more than a finite number of changes of sign. Let u_k $(k = 0, 1, 2, ...)$ be changes of sign of $F(u)$ of the form $(-+)$ (as u increases).

Theorem 1. *Self-similar solutions (s.s.) of (1) of the form $u(\xi) = u(x - \omega t)$, defined for all ξ, single-valued and having discontinuities only of the first kind, but in number not fewer than two, exist only for $\omega_k = u_k$.*

Moreover, for each such ω_k there is an infinite set of s.s. with the stated properties.

Among the indicated solutions, a special role will be played by "simple periodic" solutions, in which each period contains one discontinuity (Fig. 1).

Fig. 1

Each of the indicated s.s. is a generalized solution of the Cauchy problem for (1) with some initial condition, and, as is known (see, for example, ([1,2])), if one follows the changes of this solution in a finite rectangle $\{0 \le t \le T;\ -x_0 \le x \le x_0\}$, then it changes little under a small (in a certain metric) change of the initial data and under the introduction into the right-hand side of (1) of the term μu_{xx} with small μ. We shall, however, consider stability in the region $c < \xi < d$, which may be infinite if the solution exists for all t (this depends on the properties of $F(u)$), and shall restrict its study to the introduction of u_{xx}.

Definition. An s.s. $u(\xi) = u(x - u_0 t)$ will be called **self-similarly stable** on the interval $c < \xi < d$ if for any $\varepsilon > 0$ and $\delta > 0$ there exists a $\mu_0(\varepsilon, \delta, c, d)$ such that, for any $\mu < \mu_0$, the equation

$$u_t + uu_x = \mu u_{xx} + F(u) \tag{2}$$

has an s.s. $u_\mu(\xi) = u_\mu(x - u_0 t)$ for which, for $\xi \in (c, d)$ outside the δ-neighborhoods of the discontinuity points of $u(\xi)$, the inequality $|u(\xi) - u_\mu(\xi)| \leq \varepsilon$ is satisfied.

Of all possible s.s. of (1), "really" (i.e., taking small dissipation into account) only self-similarly stable solutions exist.

Theorem 2. *Let $F(u)$ have in the interval $(u_0 - a, u_0 + a)$ only one change of sign at $u = u_0$, of the form $(-+)$, and let, for $z < a$, $F(u_0 + z) = -F(u_0 - z)$. Then, if the solution $u(\xi)$ is self-similarly stable on the open interval $c < \xi < d$, it coincides on it with some simple periodic solution.*

The proof is based on the fact that for $u_\mu(\xi) = u_\mu(x - u_0 t)$, from (2) one obtains the dynamical system

$$dX/d\xi = Y; \qquad \mu\, dY/d\xi = XY - f(X);$$

$$X = u_\mu - u_0; \qquad Y = d(u_\mu - u_0)/d\xi; \qquad F(u - u_0) = f(X),$$

which has the point $(0, 0)$ as a center, so that every a.p. of (2) is periodic, and, moreover, in each period there is only one interval of decrease.

II. The motion of a fluid in an inclined channel is described in Lagrangian coordinates by the system

$$u_t + [p(v)]_x = F(u, v); \qquad v_t - u_x = 0; \tag{3}$$

$$p(v) = \frac{g}{2} v^{-2}; \quad F(u, v) = a - \lambda u^m \left(v + \frac{2}{l}\right)^n \operatorname{sign} u; \quad a, \lambda > 0; \quad m, n \geqslant 1,$$

(see ([3], [4]), where this system is written in Eulerian coordinates).

The jump conditions following from the divergent form (3) coincide with those usually derived from physical considerations; thus, in this sense, as in gas dynamics, the transition from the Eulerian form to the Lagrangian form is legitimate not only for smooth solutions but also for discontinuous ones.

In system (3) $v = y^{-1}$, where y is the depth of the flow. Consider a.p. of the system (3): $u = u(x - \omega t)$, $v = v(x - \omega t)$. For $m = 2$, $n = 1$, some of their properties were studied in ([3]). System (3) for these solutions is transformed into

$$[p'(v) + \omega^2]\, dv/d\xi = \widetilde{F}(v); \qquad u = -\omega v + C, \tag{4}$$

where $\widetilde{F}(v) = F(-\omega v + C, v)$.

The following theorem answers the question for which ω there exists an a.p. of (3) having at least one discontinuity.

Theorem 3. 1. There is no a.p. of (3) with $\omega < 0$.

2. The set of $\omega > 0$ for which an a.p. exists coincides with the part of the half-axis $\omega > 0$ on which the function is negative

$$f(\omega) = m\omega - n(a/\lambda)^{1/m} \left[g^{1/3}\omega^{-2/3} + 2/l\right]^{-n/m-1},$$

in particular:

a) for $2n/m < 1$, admissible ω exist and are bounded above;

b) for $2n/m \geqslant 1$, the set of admissible ω is also bounded above, but may (depending on the parameters) be empty.

3. For a wide channel ($l = \infty$), $f(\omega)$ takes the form

$$f(\omega) = m\omega - n(a/\lambda)^{1/m} g^{-n/3m-1/3}\omega^{2n/3m+2/3};$$

a) for $2n/m < 1$, the set of admissible ω satisfies the inequality

$$0 < \omega < \omega_0 = \left[\frac{n}{m}\left(\frac{a}{\lambda}\right)^{1/m} g^{-n/3m-1/3}\right]^{3/\alpha}; \qquad \alpha = 1 - \frac{2n}{3m};$$

b) for $2n/m > 1$, admissible ω are such that $\omega > \omega_0$, where ω_0 is as in a);

c) for $2n/m = 1$ (the Chézy case, see ([3])), all ω are admissible if

$$2 - (a/\lambda)^{1/2n} g^{-1/2} < 0,$$

and there are no admissible ω at all if

Fig. 2 and Fig. 3

Figure 2: Fig. 2 and Fig. 3

$$2 - (a/\lambda)^{1/2n} g^{-1/2} \geqslant 0.$$

The proof reduces to finding, for $\omega > 0$, such a $v_0(\omega)$ that

$$p'(v_0) + \omega^2 = 0,$$

and then, from $v_0(\omega)$, such a $C = C_0(\omega)$ that $\widetilde{F}(v)$ has a root at $v = v_0$, and then to investigating the sign of $\widetilde{F}'(v_0)$.

Remark 1. In cases a) of items 2 and 3, the maximum possible wave height exists for all admissible ω, and an upper estimate for it can be written as an explicit function of the parameters.

In cases b) of item 2 and b), c) of item 3, the maximum possible wave height is not bounded above.

Remark 2. The quantity n/m qualitatively affects the properties of a.p.; moreover, the frequently used Chézy formula is an exceptional

limiting case. Taking into account the empirical character of m and n, the question arises of the possibility of determining, to some degree, m and n from the qualitative behavior of the self-similar solution.

III. Similarly to the case of one equation, alongside (3) let us consider the system with a dissipative term

$$u_t + [p(v)]_x = F(u, v) + \mu u_{xx}; \qquad v_t - u_x = 0. \tag{5}$$

For solutions $v_\mu(\xi) = v_\mu(x - \omega t)$ from (5) we obtain

$$[p'(v) + \omega^2]\, dv/d\xi = F(v) - \omega\mu\, d^2v/d\xi^2. \tag{6}$$

For admissible $\omega > 0$ one can, as in the preceding item, find v_0 and C_0. Setting $v - v_0 = X;\ d(v - v_0)/d\xi = Y;\ F(v) = f(X);\ p'(v)+$

Fig. 2 **Fig. 3**

$$+\omega^2 = \varphi(X),$$

we have for (5) the dynamical system

$$dX/d\xi = Y; \qquad \omega\mu\, dY/d\xi = f(X) - \varphi(X)Y, \tag{7}$$

where the graphs of $f(X)$ and $\varphi(X)$ are shown in Fig. 2. The functions $f(X)$ and $\varphi(X)$ depend on ω.

The direction field for (7) is shown in Fig. 3. The point $(0,0)$ is a focus or a center. The investigation of the phase portrait of (7) is a very interesting problem. If, say, it turned out that the point $(0,0)$ is surrounded by a limit cycle, whose extreme left point for all possible (admissible) ω and $\mu \to 0$ does not come infinitely close to the line $X = -v_0$, then this would indicate the existence of a maximum height of waves developing from an undisturbed flow, even in cases b) of item 2 and b), c) of item 3 of Theorem 3.

Let us approximate, in a small interval $[-h, h]$, $f(X)$ and $\varphi(X)$ in (7) by linear functions $f(X) = lX$; $\varphi(X) = kX$. Then, instead of (7), for the strip $-h \leq X \leq h$ of the phase plane we shall approximately have

$$dX/d\xi = Y; \qquad \omega\mu\, dY/d\xi = lX - kXY. \tag{8}$$

For (8) the point $(0,0)$ is a center. Reasoning of the same kind as in Theorem 2 leads to the conclusion that only simple periodic solutions "really" exist (i.e., are self-similarly stable).

Since (8) approximates (7) only for small h, this conclusion, as applied to (7), is valid (and even then only approximately) only for waves of small amplitude. Its validity for waves of arbitrary amplitude is not excluded, but requires further investigation.

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Received
20 XI 1965

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Note: Figure translations are in progress. See original paper for figures.